MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XLII.

MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY ADDITIONAL

PAPERS AND SOLUTIONS

NOT PUBLISHED IN THE "EDUCATIONAL TIMES."

AND

AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR
OF THE
GENERAL MEDICAL COUNCIL.

VOL. XLII.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1885.

*** Of this series there have now been published forty-two Volumes, each of which contains, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprises contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

LIST OF CONTRIBUTORS.

ALDIS, J. S., M.A.; H.M. Inspector of Schools.
ALLEN, Rev. A. J. C., M.A.; St. Peter's Coll., Camb.
ALLMAN, Professor GEO. J., LL.D.; Galway.
ANDERSON, ALEX., B.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor; Pesaro.
BALL, ROBT. STAWELL, LL.D., F.R.S.; Professor of Astronomy in the University of Dublin.
BASU, SATISH CHANDRA; Presid. Coll., Calcutta.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BAYLISS, GEORGE, M.A.; Kenilworth.
BELTRAMI, Professor; University of Pisa.
BERG, F. J. VAN DEN; Professor of Mathematics in the Polytechnic School, Delft.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BHUT, ATH BIGAH, M.A.; Delhi.
BICKERDIKE, C.; Allerton Bywater.
BIDDLE, D.; Gough H., Kingston-on-Thames.
BIRCH, Rev. J. G., M.A.; London.
BLACKWOOD, ELIZABETH; Boulogne.
BLYTHE, W. H., B.A.; Egham.
BORCHARDT, Dr. C. W.; Victoria Strasse, Berlin.
BOSANQUET, R. H. M., M.A.; Fellow of St. John's College, Oxford.
BOURNE, C. W., M.A.; Bedford County School.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof. COLIN; Andersonian Univ. Glasgow.
BUCHHEIM, A., M.A., Ph.D.; Schol. New Coll., Oxf.
BUCK, EDWARD, M.A.; Univ. Coll., Bristol.
BURNSIDE, W. S., M.A.; Professor of Mathematics in the University of Dublin.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., B.A.; Endsleigh Gardens, London.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge, Member of the Institute of France, &c.
CHAKRAVARTI, BYOMAKESA, M.A.; Professor of Mathematics, Calcutta.
CHASE, PLINY EARLE, LL.D.; Professor of Philosophy in Haverford College.
CLARKE, Colonel A. R., C.B., F.R.S.; Hastings.
COATES, W. M., B.A.; Trinity College, Dublin.
COCHEZ, Professor; Paris.
COCKLE, Sir JAMES, M.A., F.R.S.; Bayswater.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrew's.
CONSTABLE, S.; Swinford Rectory, Mayo.
COTTERILL, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
CROFTON, M. W., B.A., F.R.S.; Prof. of Math. and Mech. in the R. M. Acad., Woolwich.
CULVERWELL, E. P., B.A.; Sch. of Trin. Coll. Dubl.
CURTIS, ARTHUR HILL, LL.D., D.Sc.; Dublin.
DARBOUX, Professor; Paris.
DAVIS, J. G., M.A.; Abingdon
DAVIS, R. F., B.A.; Wandsworth Common.
DAWSON, H. G., B.A.; Baymount, Dublin.
DAY, Rev. H. G., M.A.; Richmond Terr., Brighton.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DICK, G. R., M.A.; Fellow of Caius Coll., Camb.
DOBSON, T., B.A.; Hexham Grammar School.
DROZ, Prof. ARNOLD, M.A.; Porrentruy, Berne.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTWOOD, G., M.A.; Saxonville, Massachusetts.
EASTON, BELLE; Lockport, New York.
EDWARD, J., M.A.; Head Mas'., Aberdeen Coll.
EDWARDES, DAVID; Erith Villas, Erith, Kent.
ELLIOTT, E. B., M.A.; Fell. Queen's Coll., Oxon.
ELLIS, ALEXANDER J., F.R.S.; Kensington.
EMTAGE, W. T. A.; Pembroke Coll., Oxford.
ESCOTT, ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.
ESSENNELL, EMMA; Coventry.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in Queen's College, Belfast.
EVANS, Professor, M.A.; Lockport, New York.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FINCH, T. H., B.A.; Trinity College, Dublin.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FRANKLIN, CHRISTINE LADD, M.A.; Prof. of Nat. Sci. and Math., Union Springs, New York.
FUORTÈS, E.; University of Naples.
GALBRAITH, Rev. J., M.A., Fell. Trin. Coll., Dublin.
GALE, KATE K.; Worcester Park, Surrey.
GALLATLY, W., B.A.; Earl's Court, London.
GALLIERS, Rev. I., M.A.; Kirkstead Rectory, Norwich.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., B.A.; Stud. of Ch. Ch., Oxford.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GOLDENBERG, Professor, M.A.; Moscow.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENSTREET, W. J., B.A.; Framlingham.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Harold, B.A.; King's Coll., Cambridge.
HARLEY, Rev. R., F.R.S.; Huddersfield College.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HARRIS, J. R., M.A.; Clare College, Cambridge.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; The Grove, Hammersmith.
HERBERT, A., M.A.; King Alfred's Sch., Wantage.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOLT, J. R., M.A.; Trinity College, Dublin.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
INGLEBY, C. M., M.A., LL.D.; London.
JENKINS, MORGAN, M.A.; London.
JOHNSON, W. E., B.A.; King's Coll., Cambridge.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, SWIFT; Trin. Coll., Dublin.
JONES, H. S., M.A.; Llanelly.
JONES, L. W., B.A.; Merton College, Oxford.
KEALY, J. A., M.A.; Wilmington, Delaware.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rect.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
KNOWLES, R., B.A., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
LACHLAN, R., B.A.; Lewisham.
LARMOR, J., M.A.; Queen's College, Galway.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAX, W. G., B.A.; Trinity College, Cambridge.
LEIDHOLD, R., M.A.; Finsbury Park.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, A., D.Sc., F.R.S.E.; Ontario.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MACMAHON, Capt. P. A.; R. M. Academy.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.

McAlister, Donald, M.A., D.Sc.; Cambridge.
McCay, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
McClelland, W. J., B.A.; Prin. of Santry School.
McColl, H., B.A.; 73, Rue Sibliquin, Boulogne.
McDowell, J., M.A.; Pembroke Coll., Camb.
McIntosh, Alex., B.A.; Bedford Row, London.
McLeod, J., M.A.; R.M. Academy, Woolwich.
McVicker, C. E., B.A.; Trinity Coll., Dublin.
Malet, Prof., M.A.; Queen's Coll., Cork.
Mannheim, M.; Prof. à l'Ecole Polytech., Paris.
Marks, Sarah; Cambridge Street, Hyde Park.
Martin, Artemas, M.A., Ph.D.; Editor & Printer of *Math. Visitor & Math. Mag.*, Erie, Pa.
Martin, Rev. H., D.D., M.A.; Edinburgh.
Mathews, G. B., B.A.; Univ. Coll., N. Wales.
Matz, Prof., M.A.; King's Mountain, Carolina.
Mee, W. M., B.A.; Belturbet.
Merrifield, J., LL.D., F.R.A.S.; Plymouth.
Merriman, Mansfield, M.A.; Yale College.
Meyer, Mary S.; Girton College, Cambridge.
Miller, W. J. C., B.A., (Editor); The Paragon, Richmond-on-Thames.
Minchin, G.M., M.A.; Prof. in Cooper's Hill Coll.
Mitcheson, T., B.A., L.C.P.; City of London Sch.
Monck, Henry Stanley, M.A.; Prof. of Moral Philosophy in the University of Dublin.
Moncourt, Professor; Paris.
Moon, Robert, M.A.; Ex-Fell. Qu. Coll., Camb.
Moore, H. K., B.A.; Trin. Coll., Dublin.
Morel, Professor; Paris.
Morgan, C., B.A.; Salisbury School.
Morley, Frank, B.A.; Bath Coll., Bath.
Morley, T., L.C.P.; Bromley, Kent.
Morrice, G. G., B.A.; The Hall, Salisbury.
Moulton, J. F., M.A.; Fell. of Ch. Coll., Camb.
Muir, Thomas, M.A., F.R.S.E.; Bishopton.
Mukhopadhyay, Asutosh, M.A.; Bhowanipore.
Nash, A. M., M.A.; Prof. in Pres. Coll., Calcutta.
Nelson, R. J., M.A.; Naval School, London.
Newcomb, Prof. Simon, M.A.; Washington.
Nicolls, W., B.A.; St. Peter's Coll., Camb.
Openshaw, Rev. T. W., M.A.; Clifton.
O'Regan, John; New Street, Limerick.
Orchard, H. L., M.A., L.C.P.; Burnham.
Owen, J. A., B.Sc.; Tennyson St., Liverpool.
Panton, A. W., M.A.; Fell. of Trin. Coll., Dublin.
Pendlebury, Rev. C., M.A.; London.
Perryman, W.; Carbrook, Sheffield.
Phillips, F. B. W.; Balliol College, Oxford.
Pillai, C. K., M.A.; Trichy, Madras.
Pirie, A., M.A.; University of St. Andrew's.
Polignac, Prince Camille de; Paris.
Pollexfen, H., B.A.; Windermere College.
Potter, J., B.A.; Richmond-on-Thames.
Prudden, Frances E.; Lockport, New York.
Purser, Prof. F., M.A.; Queen's College, Belfast.
Putnam, K. S., M.A.; Rome, New York.
Rau, B. Hanumanta, B.A.; Head Master of the Normal School, Madras.
Rawson, Robert; Havant, Hants.
Raymond, E. Lancelot, B.A., Surbiton.
Ray, Prof. Saradaranjan, M.A.; Dacca Coll., Bengal.
Read, H. T., M.A.; Brasenose Coll., Oxford.
Reeves, G. M., M.A.; Lee, Kent.
Renshaw, S. A.; Nottingham.
Reynolds, B., M.A.; Notting Hill, London.
Richardson, Rev. G., M.A.; Winchester.
Rippin, Charles R., M.A.; Woolwich Common.
Roberts, R. A., M.A.; Schol. of Trin. Coll., Dublin.
Roberts, S., M.A., F.R.S.; Tufnell Park, London.
Roberts, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
Roberts, W. R., M.A.; Ex-Sch. of Trin. Coll., Dub.
Robson, H. C., B.A.; Sidney Sussex Coll., Cam.
Rosenthal, L. H.; Scholar of Trin. Coll., Dublin.
Roy, Prof. Kaliprasanna, M.A.; St. John's Coll., Agra.
Rücker, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
Ruggero, Simonelli; Università di Roma.
Russell, J. W., M.A.; Merton Coll., Oxford.
Russell, R., B.A.; Trinity College, Dublin.
Rutter, Edward; Sunderland.
Salmon, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
Sampson, C. H., M.A.; Balliol College, Oxford.
Sanders, J. B.; Bloomington, Indiana.
Sanderson, Rev. T. J., M.A.; Royston, Cambs.
Sarkar, Nilkantha, M.A.; Calcutta.
Savage, Thomas, M.A.; Fell. Pemb. Coll., Camb.
Scheffer, Professor; Mercersbury Coll., Pa.
Scott, A. W., M.A.; St. David's Coll., Lampeter.
Scott, Charlotte A., B.Sc.; Manchester.
Scott, R. F., M.A.; Fell. St. John's Coll., Camb.
Serret, Professor; Paris.
Sharp, W. J. C., M.A.; Hill Street, London.
Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham.
Shepherd, Rev. A. J. P., B.A.; Fellow of Queen's College, Oxford.
Simmons, Rev. T.C., M.A.; Christ's Coll., Brecon.
Siverly, Walter; Oil City, Pennsylvania.
Smith, C., M.A.; Sidney Sussex Coll., Camb.
Stabenow, H., M.A.; New York.
Steggall, Prof. J. E. A., M.A.; Clifton.
Stein, A.; Venice.
Stephen, St. John, B.A.; Caius Coll., Cambridge.
Stewart, H., M.A.; Framlingham, Suffolk.
Storr, G. G., B.A.; Blackburn.
Swift, C. A., B.A.; Grammar Sch., Weybridge.
Sylvester, J. J., D.C.L., F.R.S.; Professor of Mathematics in the University of Oxford, Member of the Institute of France, &c.
Symons, E. W., M.A.; Fell. St. John's Coll., Oxon.
Tait, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
Tanner, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
Tarleton, F. A., M.A.; Fell. Trin. Coll., Dub.
Taylor, Rev. C., D.D.; Master of St. John's College, Cambridge.
Taylor, H. M., M.A.; Fell. Trin. Coll., Camb.
Taylor, W. W., M.A.; Ripon Grammar School.
Tebay, Septimus, B.A.; Farnworth, Bolton.
Terry, Rev. T. R., M.A., Fell. Magd. Coll., Oxon.
Thomas, Rev. D., M.A.; Garsington Rect., Oxford.
Thomson, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Camb.; Brinkley Rectory, Newmarket.
Tirelli, Dr. Francesco; Univ. di Roma.
Todhunter, Isaac, F.R.S.; Cambridge.
Torelli, Gabriel; University of Naples.
Torry, Rev. A. F., M.A.; St. John's Coll., Camb.
Traill, Anthony, M.A., M.D.; Fellow and Tutor of Trinity College, Dublin.
Trowbridge, David; Waterburgh, New York.
Tucker, R., M.A.; Mathematical Master in University College School, London.
Turrell, I. H.; Cumminsville, Ohio.
Turriff, George, M.A.; Aberdeen.
Vincenzo, Jacobini; Università di Roma.
Vose, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
Walenn, W. H.; Mem. Phys. Society, London.
Walker, J. J., M.A., F.R.S.; Hampstead.
Walmsley, J., B.A.; Eccles, Manchester.
Warburton-White, R., B.A., Salisbury.
Warren, R., M.A.; Trinity College, Dublin.
Watherston, Rev. A. L., M.A.; Bowdon.
Watson, Rev. H. W.; Ex-Fell. Trin. Coll., Camb.
Wertsch, Fr.; Weimar.
White, J. R., B.A.; Worcester Coll., Oxford.
White, Rev. J., M.A.; Cowley College, Oxford.
Whiteside, G., M.A.; Eccleston, Lancashire.
Whitworth, Rev. W. A., M.A.; Fellow of St. John's Coll., Camb.; Hammersmith.
Wickersham, D.; Clinton Co., Ohio.
Wilkins, W.; Scholar of Trin. Coll., Dublin.
Williamson, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
Wilson, J. M., M.A.; Head-master, Clifton Coll.
Wilson, Rev. J., M.A.; Rect. Bannockburn Acad.
Wilson, Rev. J. R., M.A.; Royston, Cambs.
Woodcock, T., B.A.; Wellington, Salop.
Wolstenholme, Rev. J., M.A., Sc.D.; Professor of Mathematics in Cooper's Hill College.
Woolhouse, W. S. B., F.R.A.S., &c.; London
Wright, Dr. S. H., M.A.; Penn Yan, New York.
Wright, W. E., B.A.; Herne Hill.
Young, John, B.A.; Academy, Londonderry.

CONTENTS.

Mathematical Papers, &c.

Questions Solved.

2Y

Appendix.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7706. (By the late Professor CLIFFORD, F.R.S.)—What conditions must be fulfilled in order that the centre of pressure of a triangle wholly submerged in water may be at the intersection of perpendiculars?

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

When the vertices A, B, C of a triangle are sunk in a homogeneous liquid to depth h_1, h_2, h_3, respectively, then x, y, z, the coordinates of the centre of pressure of the triangle referred to the sides as axes, are given by the equations :—

$$x = \frac{p_1}{4}\left\{1+\frac{h_1}{h_1+h_2+h_3}\right\}, \quad y = \frac{p_2}{4}\left\{1+\frac{h_2}{h_1+h_2+h_3}\right\},$$

$$z = \frac{p_3}{4}\left\{1+\frac{h_3}{h_1+h_2+h_3}\right\}$$

(see *Messenger of Mathematics*, No. 8, 1864); but the coordinates of the intersection of the perpendiculars p_1, p_2, p_3 are

$$x = p_1 - \frac{b\cos A}{\sin B} = p_1\left(1-\frac{\cos A}{\sin B\sin C}\right),$$

hence we have $p_1\left(1-\dfrac{\cos A}{\sin B\sin C}\right) = \dfrac{p_1}{4}\left(1+\dfrac{h_1}{h_1+h_2+h_3}\right)$,

$$\therefore\quad \frac{h_1}{h_1+h_2+h_3} = 3-\frac{4\cos A}{\sin B\sin C} = 3+\frac{4\cos(B+C)}{\sin B\sin C} = 4\cot B\cot C - 1,$$

and two similar equations;

$\therefore\ h_1 : h_2 : h_3 :: 4\cot B\cot C-1 : 4\cot A\cot C-1 : 4\cot A\cot B-1.$

[For a discussion of this problem, and several other applications of the above formulæ for x, y, z, see *Messenger of Mathematics*, New Series, No. 12, 1872, where it is shown that, if the vertices of a triangle be sunk to depths h_1, h_2, h_3, and therefore the mid-points of the sides to depths

$H_1 = (h_2+h_3)$, $H_2 = \frac{1}{3}(h_1+h_3)$, $H = \frac{1}{3}(h_1+h_2)$, when the centre of pressure coincides with:

	Condition.
(1) Centre of gravity	$\}\ H_1 = H_2 = H_3$,
(2) Centre of circumscribing circle	$\}\ H_1 : H_2 : H_3 = \tan A : \tan B : \tan C$,
(3) Centre of inscribed circle	$\}\ H_1 : H_2 : H_3 = \cot \frac{1}{2}A : \cot \frac{1}{2}B : \cot \frac{1}{2}C$,
(4) Intersection of the three perpendiculars from vertices on opposite sides	$\}\ H_1 : H_2 : H_3 = 1 - 2\cot B \cot C : 1 - 2\cot A \cot C : 1 - 2\cot A \cot B$,
(5) Centre of inscribed square (alongside a)	$\}\ H_1 : H_2 : H_3 = 1 : \cot B : \cot C$,
(6) Centre of the nine-point circle	$\}\ H_1 : H_2 : H_3 = \sin 2A : \sin 2B : \sin 2C$;

hence it follows that, if a triangle be so immersed that its centre of pressure coincides with the intersection of the three perpendiculars from vertices on the opposite sides, then the centre of pressure of the triangle obtained from the given one by joining the middle points of its sides will coincide with the centre of the circle circumscribed to this derived triangle, and the centre of pressure of the triangle similarly derived from it will coincide with the centre of the nine-point circle of this last derived triangle.]

7708. (By Professor TOWNSEND, F.R.S.)—A thin uniform spherical cap being supposed to attract according to the law of the inverse fifth power of the distance a material particle situated anywhere on the surface of the sphere; show that, for every position of the particle, the attraction (a) passes through the vertex of the cone which envelopes the sphere along the rim of the cap, (b) varies directly as the radial distance from the vertex of the cone, and inversely as the cube of the perpendicular distance from the base of the cap.

Solution by the PROPOSER.

Two pairs of planes inclined at elementary angles, one pair passing through the line of connection of the particle with the vertex of the cone, and the other pair passing through its polar with respect to the sphere, which latter lies of course in the rim-plane or base of the cap, will intercept in the mass of the cap a pair of quadrilateral elements whose attractions on the particle for the law of the inverse fifth power of the distance are easily seen to compound a resultant directed to the vertex of the cone; and, as the entire mass of the cap may manifestly be exhausted by pairs of such elements, therefore, &c., as regards the first part of the property.

As regards the second part. The integration based on the preceding method of division of the cap into elements leads to the result that the entire attraction of the cap on the particle $= \dfrac{m(a+b)b}{16a^5} \cdot \dfrac{r}{p^3}$; where a is the radius of the sphere, m the mass of the cap, b the distance of its base from the centre of the sphere, r the distance of the particle from the vertex of the cone, and p its distance from the base of the cap. And therefore, &c., as regards the second part also.

7709. (By Professor MINCHIN, M.A.)—A cylindrical bar of isotropic material is subject to uniform intensity of pressure over its *curved* surface; prove that, if M denote the "modulus of cylindric squeeze," while k and μ denote the resistances to cubical squeeze and to distortion, then $M = \frac{9k\mu}{3k+4\mu}$.

Solution by J. BRILL, B.A.; Prof. MATZ, M.A.; *and others.*

Taking the axis of z along the axis of the cylinder, measuring z from the middle point of the axis, and assuming $u = -ax$, $v = -ay$, $w = cz$, we have

$$N_1 = N_2 = (k - \tfrac{2}{3}\mu)(c-2a) - 2\mu a,$$
$$N_3 = (k - \tfrac{2}{3}\mu)(c-2a) + 2\mu c.$$

Since there is no force on the ends of the bar, we must have $N_3 = 0$,

therefore $\quad 0 = (k-\tfrac{2}{3}\mu)(c-2a)+2\mu c; \quad$ therefore $c = 2a\,\dfrac{3k-2\mu}{3k+4\mu}$;

$$\therefore \quad M = \frac{[(k-\frac{2}{3}\mu)(c-2a)-2\mu a]}{-2a} = (k-\tfrac{2}{3}\mu)\left(1-\frac{3k-2\mu}{3k+4\mu}\right)+\mu$$
$$= \tfrac{1}{3}\left\{3k+\mu-\frac{(3k-2\mu)^2}{3k+4\mu}\right\} = \frac{9k\mu}{3k+4\mu}.$$

6251. (By the EDITOR.)—One of the diagonals of a regular quindecagon is drawn at random, and then the process is repeated; show that (1) the probability of the chosen diagonals being such as cross within the perimeter is $\frac{91}{267}$ if the two must be distinct, and $\frac{91}{270}$ if the second may be identical with the first; (2) the like probabilities for a regular $(2n+1)$-gon are $\frac{1}{3}(2n^2-n)$ divided, in the two cases respectively, by $[(2n+1)(n-1)-1]$ or $[(2n+1)(n-1)]$; and hence (3) the chance of two random chords meeting within a circle is $\frac{90}{270}$ or $\frac{1}{3}$.

Solution by D. BIDDLE, *and the* PROPOSER.

1. From each of the 15 angles there are 12 diagonals, or in all $180 \div 2 = 90$. The 6 which cross one-half of the figure are similar to the other 6. Taking the one 6 in order, we find that they are crossed by diagonals as follows:— I. by 12; II. by $(12-1)\times 2 = 22$; III. by $(12-2)\times 3 = 30$; IV. by $(12-3)\times 4 = 36$; V. by $(12-4)\times 5 = 40$; VI. by $(12-5)\times 6 = 42$. There are accordingly 6 differently circumstanced sets of diagonals, and the probability as to which of these is selected in the first diagonal drawn is $\frac{1}{6}$. For the second, there are 89 to choose from if the two must be distinct, 90 if the choice may be identical; and of these a particular number, as specified above, cross the first, but the rest do not. Hence the total probability that the chosen diagonals will cross within the perimeter is $(12+22+30+36+40+42)$, or 182, divided by $6 \,.\, 89$ or $6 \,.\, 90$ in the two cases respectively, which give the stated results.

2. For a regular $(2n+1)$-gon, there are $(n-1)$ different cases, all equally probable, since the first diagonal may cut off 1, 2, 3, ..., $(n-1)$ corners of

the polygon; and the probability that the second diagonal will cross the first is, in each of these several cases,

$$(2n-2),\ 2\,(2n-3),\ 3\,(2n-4),\ \ldots,\ (n-1)\,(2n-n),$$

divided (N being the total number of diagonals) by N − 1 or N, according as we *ex*clude or *in*clude the first diagonal in the second drawing; and, as $N = (2n+1)(n-1)$, this leads at once to the results stated in the question.

3. If we suppose n to increase without limit, the polygon becomes, in the limit, a circle whereof the diagonals are chords, and the limit of the fraction that measures the probability is $\frac{1}{3}$.

Otherwise.—The crossing or not, within the circumference, of two random chords of a circle, will be governed by the same law. Let C be the circumference of a given circle, and A the arc of a given chord in it. Then $\dfrac{2A\,(C-A)}{C^2}$ will represent the proportion of chords crossing the given chord (the 2 in the numerator arising from C^2 in the denominator being divided by 2, to prevent our taking chords twice over, that is, from both ends). And we can form a series with $A = 0$ at one extremity, and $A = \frac{1}{2}C$ at the other, a diameter being the longest chord possible. In the former $\dfrac{2A\,(C-A)}{C^2} = 0$, in the latter $\frac{1}{2}$. Let $C = 1$, then our formula becomes $2A\,(1-A)$, and, since $1-A = A$ in the first term of the descending series, we can construct a new formula, viz., $2\,(\frac{1}{2}-b)\,(\frac{1}{2}+b)$, where b = the portion of C deducted from one factor and added to the other, to form any given term of the series; and this simplified is $2\,(\frac{1}{4}-b^2)$. Moreover, b^2 gradually increases from 0 to $\frac{1}{4}$, that is $(\frac{1}{2})^2$; and as $2\,(\frac{1}{4}-b^2)$ represents the probability, as to being crossed, of the chord of any given arc, we need only sum the series and divide by the number of terms, to obtain the average probability. Now the average of such terms, where each consists of two square numbers, equal at the beginning, but one constant and the other gradually diminishing to zero, is equivalent to the mean-section-area of the space intervening between a hollow cylinder, of given height and internal diameter, and a cone of the same height, whose base just fits the cylinder. Let the height = 1, and the base = $\frac{1}{4}$, then the area of the cone = $\frac{1}{12}\,.\,1^3$, and its mean transverse section-area = $\frac{1}{12}\,.\,1^2$ = mean value of b^2 in above formula. Hence $2\,(\frac{1}{4}-b^2)$ is on the average $\frac{1}{3}$, the probability stated in the question.

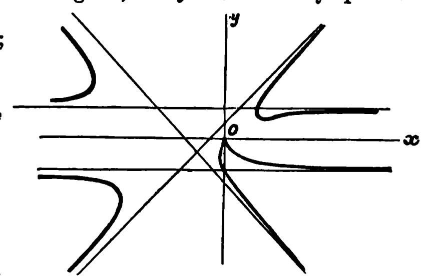
6607. (By Christine Ladd Franklin, B.A.)—Trace the curve whose equation is $(y^2-ax)^2 = y^2\,(x^2-ay)$, and test the correctness of the form by applying Descartes' rule of signs to determine limits to the number of points of intersection of the curve by lines parallel to the axis of y.

Solution by the Rev. T. C. Simmons, M.A.

The given equation $\equiv y^4 - x^2y^2 - 2ay^2x + ay^3 + a^2x^2 = 0$, whence

$$x = \left[-a \pm (y^2+ay-a^3y^{-1})^{\frac{1}{2}}\right] / \,(1-a^2y^{-2})$$
$$= -y - \tfrac{3}{2}a - \tfrac{5}{8}a^2y^{-1} + \ldots \text{ or } y - \tfrac{1}{2}a + \tfrac{5}{8}a^2y^{-1} + \ldots,$$

whence $x = -y-\frac{3}{2}a$ and $x = y-\frac{1}{2}a$ are asymptotes, having the curves situated on the sides shown in the figure; also $y = \pm a$ are asymptotes.



Near the origin

$y^3+ax^2 = 0$ approximately;

also

$$\frac{dy}{dx} = \frac{2y^2x+2ay^2-2a^2x}{4y^3-2yx^2+3ay^2-4ayx}$$

$= 0$ at intersection of (1) with $y^2x+ay^2-a^2x = 0$, which is the same as the intersection of $xy=a^2$ with

$$y^2-ax+ay = 0,$$

giving $x=\frac{4}{3}a$, $y=\frac{3}{4}a$ nearly.

The value of x shows that y^4+ay^3 must be $> a^3y$, so that y cannot lie between 0 and $\frac{3}{4}a$.

At the points $(0, -a)$, (a, a), $dy : dx$ is -2 and 2 respectively. To find approximately the contour of the curve towards the left, putting $y = 2a$, x becomes $-\frac{13}{3}a$ nearly and $dy : dx = -\frac{9}{2}$; and putting $y = -2a$, x becomes $-\frac{10}{3}a$ nearly and $dy : dx = -\frac{6}{5}$; hence the figure is as annexed.

[As to the latter part of the question, writing the equation in the form $y^4+ay^3-(x^2+2ax)y^2+a^2x^2 = 0$, we see that, by Descartes' rule of signs, when x^2+2ax is positive, there cannot be more than two positive nor more than two negative values of y; and that, when x^2+2ax is negative, that is, when x lies between 0 and $-2a$, there cannot be *any* such positive values, and not more than two negative ones.]

7710. (By Professor Cochez.)—Parmi les courbes planes uniformement pesantes de même longueur, passant par deux points fixes, quelle est celle dont le centre de gravité est le plus bas?

Solution by Arthur Hill Curtis, LL.D., D.Sc.

When any system of heavy particles are in equilibrium under the action of gravity alone, the position of stable equilibrium is such that the centre of gravity of the system is the lowest possible; as a particular case, a string of constant or variable density throughout will assume, as its position of equilibrium, the curve which satisfies this condition; the curve required in the question is consequently the catenary.

The Symmedian-Point Axis of a System of Triangles.

By R. Tucker, M.A.

Through the angular points of the triangle ABC straight lines are drawn parallel to the opposite sides forming the triangle A′B′C′: the process is repeated indefinitely. This is the system herein considered.

Let AD, BE, CF be the perpendiculars meeting in the orthocentre P, and let D′, E′, F′ be the mid-points of the sides. Now take on the sides $A\alpha = CD$, $C\beta = AE$, $B\gamma = AF$, then $A\alpha$, $B\beta$, $C\gamma$ intersect in a point π. Hence, by an application of § 11 of NEUBERG's pamphlet *Médianes Anti-parallèles* (or *Educational Times*, May; Note to Solution of Question 7644), we see that π is the Symmedian-point of A′B′C′: and $P\pi$ is the diameter of the Brocard Circle of that triangle. If K, K′ are the S-points of ABC, D′E′F′, it may be proved that they lie on the line $K\pi$; hence the S-points of the system of triangles under consideration lie on $K\pi$, the equation to which is $a\alpha(b^2-c^2)+b\beta(c^2-a^2)+c\gamma(a^2-b^2)=0$. This line evidently passes through the centroid (G) of the system.

Many consequences follow from this property (see *Quarterly Journal of Mathematics*, Vol. xx., No. 78). We may note here that the Brocard-circle of ABC bisects the chord $K\pi$ produced, of the circum-circle, making that circle an eight-point circle.

If perpendiculars are drawn to the sides of ABC through the points α, β, γ, they meet in a point π', such that $a\pi' = 2R(\cos A \sim \cos B \cos C)$.

If O be the circumcentre then the line OP whose equation is

$$\alpha(b^2-c^2)\cos A+\beta(c^2-a^2)\cos B+\gamma(a^2-b^2)\cos C=0$$

passes through the five points P, Nine-Point-circle's centre, G, O, π'; and $\pi\pi' = 2OO_1'$, where O_1' is mid-point of $P\pi$ (see *Quar. Jour. loc. cit.*)

M. MAURICE D'OCAGNE sends the following neat property:—

If M, any point on the circum-circle, be joined to K, then MK produced cuts the sides in C′, B′, A′, so that $\frac{3}{MK}=\frac{1}{MA'}+\frac{1}{MB'}+\frac{1}{MC'}$; *i.e.*, K is the centre of Harmonic means of the points C′, B′, A′, with respect to M. M. D'OCAGNE remarks that: "Ce théorème renferme diverses conséquences que vous verrez aisément."

7718. (By C. LEUDESDORF, M.A.)—Find the value of the determinant

$$\begin{vmatrix} a & b & b & . & . \\ b & 0 & b & . & . \\ b & b & a & . & . \\ . & . & . & . & . \end{vmatrix}$$

where the leading diagonal consists of a and zero alternately, and the other constituents are each b; the determinant having n rows.

Solution by GEORGE HEPPEL, M.A.

Let D_m stand for the value of a determinant of m rows. Then, taking m even and odd, and in each case subtracting the last column but one from the last, and afterwards the last row but one from the last, we obtain immediately the relations

$$D_{2n+2}=(a-2b)D_{2n+1}-(a-b)^2D_{2n},\qquad D_{2n+1}=(a-2b)D_{2n}-b^2D_{2n-1}.$$

By putting $n=1, 2, 3$, &c., we arrive at a law, easily verified by induction,

$$D_{2n+1}=(-1)^{n+1}b^n(a-b)^n[(n-1)a-2nb],$$
$$D_{2n+2}=(-1)^{n}b^{n+1}(a-b)^n[na-(2n+1)b].$$

7667. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Prove that, if n be a positive integer,

$$\frac{1+\frac{1}{2}\sin^2\theta+\frac{1.3}{2.4}\sin^4\theta+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\sin^{2n}\theta}{1-n\frac{\cos^2\theta}{3}+\frac{n(n-1)}{2!}\frac{\cos^4\theta}{5}-\frac{n(n-1)(n-2)}{3!}\frac{\cos^6\theta}{7}+\ldots+(-1)^n\frac{\cos^{2n}\theta}{2n+1}}$$

$$=\frac{1+(\frac{1}{2})^2+\left(\frac{1.3}{2.4}\right)^2+\ldots+\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)^2}{1-n.\frac{1}{2.3}+\frac{n(n-1)}{2!}.\frac{1.3}{2.4.5}-\ldots+(-1)^n\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n(2n+1)}}$$

$$=\frac{3\,.\,5\ldots(2n+1)}{2\,.\,4\,.\,6\ldots 2n}.$$

Solution by G. HEPPEL, M.A.; E. BUCK, M.A.; *and others.*

Of the three members of the identity to be established, call the first $\frac{h}{k}$, the second $\frac{l}{m}$, and the third p; then we have

$$\frac{h}{k}=\frac{-h\cos\theta}{-k\cos\theta}=\frac{p\int\sin^{2n+1}\theta\,d\theta}{\int\sin\theta\,(1-\cos^2\theta)^n\,d\theta}=p,$$

also
$$\frac{l}{m}=\frac{\int_0^{\frac{1}{2}\pi}p\sin^{2n+1}d\theta}{\int_0^{\frac{1}{2}\pi}\sin\theta\,(1-\cos^2\theta)^n\,d\theta}=p.$$

7654. (By ASÚTOSH MUKHOPÂDHYÂY, M.A.)—A magnetic needle is free to revolve in a horizontal plane round a fixed point in the line joining its poles; if it is acted on by an indefinitely extending vertical galvanic current: find (1) the positions of equilibrium; (2) the cases wherein there is no position of equilibrium; and (3) when the positions of stable and unstable equilibrium are directly opposed.

Note by the EDITOR.

A full discussion of this Question, with the results sought—too long for our pages—is to be found in Dr. CURTIS's paper, published in The *Oxford, Cambridge, and Dublin Messenger of Mathematics* (No. 8, 1864).

7604. (By Rev. T. P. KIRKMAN, M.A., F.R.S.) —If a_p be the number of the p-aces on a triangular-faced n-edron, prove that $n-S_p\,(p-4)\,a_p$ is a cube.

Solution by EDWARD BUCK, M.A.

In any polyedron, if f_p be the number of p-gons (faces with p-sides),

$$e = \text{total number of edges} = \tfrac{1}{2}[3f_3+4f_4+5f_5+\&c.] = \tfrac{1}{2}[3a_3+4a_4+5a_5+\&c.]$$
$$= \tfrac{1}{4}[3(a_3+f_3)+4(a_4+f_4)+\&c.],$$

S = No. of summits $= a_3 + a_4 + a_5 + \&c.$, f = No. of faces $= f_3 + f_4 + f_5 + \&c.$,
$S + f = (a_3 + f_3) + a_4 + f_4 + \&c. = e + 2 = 2 + \frac{1}{4}[3(a_3 + f_3) + 4(a_4 + f_4) + \&c.]$,
therefore $$a_3 + f_3 = 8 + a_5 + f_5 + 2(a_6 + f_6).$$

If $f_5 = f_6 = 0$, $f = f_3 = 8 - a_3 + a_5 + 2a_6 + \&c. = 8 + \Sigma(p-4)\,a_p$, therefore $f - \Sigma(p-4)\,a_p = 8$, which proves the proposition.

7729. (By B. REYNOLDS, M.A.)—Show that the number of shortest routes from one corner of a chess-board to the opposite one, along the edges of the squares, is 12870.

Solution by D. BIDDLE.

The corners of squares being junctions, we may observe that from every junction, except on the two edges of the board which terminate in the goal, either of two directions may be taken; and that the number of shortest routes possible after reaching any junction, is the sum of the shortest routes possible from the two adjacent junctions over one or other of which we must next pass. Accordingly the total number of possible shortest routes from the extreme corner can be found by forming successive series of numbers, beginning with units and making the successive terms (except the first) of one series the successive differences of the next.

The following table gives the several series, and in so doing the number of shortest routes possible from every junction on the board, A being the goal:—

A	1,	1,	1,	1,	1,	1,	1,	1,
1,	2,	3,	4,	5,	6,	7,	8,	9,
1,	3,	6,	10,	15,	21,	28,	36,	45,
1,	4,	10,	20,	35,	56,	84,	120,	165,
1,	5,	15,	35,	70,	126,	210,	330,	495,
1,	6,	21,	56,	126,	252,	462,	792,	1287,
1,	7,	28,	84,	210,	462,	924,	1716,	3003,
1,	8,	36,	120,	330,	792,	1716,	3432,	6435,
1,	9,	45,	165,	495,	1287,	3003,	6435,	12870.

[Suppose we have to proceed, as quickly as possible, from O (the left-hand corner at bottom) to C (the right-hand corner at top of square). Let a denote the operation of moving horizontally over the edge of one square from left to right, and let b denote the operation of moving one step upwards. Then we must perform 8 a's and 8 b's in any order; hence the total number of ways is $(16!)\,/\,(8!\;8!)$, which reduces to 12870.]

7341. (By A. MARTIN, B.A.)—Solve the equations
$$yz(y+z-x) = a, \quad zx(z+x-y) = b, \quad xy(x+y-z) = c.$$

Solution by the Rev. T. C. SIMMONS, M.A.

[As the solution of this question given on p. 26 of Vol. 40 of our *Reprints* has the disadvantage of containing *two* cubics to be solved, and in the final cubic $x^3-px^2+qx-r=0$ it is not shown how to determine which roots belong to x, y, z respectively, the following method has been suggested as preferable.]

The three equations may be written

$$-\left(\frac{a}{xyz}+1\right)x+y+z=0,\quad x-\left(\frac{b}{xyz}+1\right)y+z=0\ \ldots\ldots(1,\,2),$$

$$x+y-\left(\frac{c}{xyz}+1\right)z=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

whence

$$\begin{vmatrix} -\left(\frac{a}{xyz}+1\right) & 1 & 1 \\ 1 & -\left(\frac{b}{xyz}+1\right) & 1 \\ 1 & 1 & -\left(\frac{c}{xyz}+1\right) \end{vmatrix}=0,$$

or

$$4(xyz)^3-(ab+bc+ca)\,xyz-abc=0.$$

Let p be any root of this equation, then in (1), (2), (3) we can find the ratios of x, y, z in the form $x=mz$, $y=nz$, therefore $mnz^3=p$, therefore z, and consequently x and y, are determined.

7358. (By the EDITOR.)—A parabola and a semi-ellipse of excentricity e have the same focus and parameter, the parabola being terminated by the minor axis of the ellipse; prove that, if the two figures revolve about their common axis,

Volume generated by parabola : Vol. gen. by ellipse $=3(1+2e-e^2)^2:8$.

Solution by the Rev. T. C. SIMMONS, M.A.

Taking a, $a(1-e^2)^{\frac{1}{2}}$ as the semi-axes of the ellipse, the volume of the semi-ellipsoid of revolution will be $V=\frac{2}{3}\pi a^3(1-e^2)$. The parabola will have $2a(1-e^2)$ for latus-rectum, and $ae+\frac{1}{2}a(1-e^2)$ for length of axis; hence the volume of the paraboloid of revolution will be

$$V'=2a\pi(1-e^2)\int_0^{\frac{1}{2}a(1+2e-e^2)}x\,dx=\tfrac{1}{4}\pi a^3(1-e^2)(1+2e-e^2)^2;\ \text{therefore, \&c.}$$

6218 & 7454. (By Professor SYLVESTER, F.R.S.)—If E_i denote

$$a_0\frac{d}{da_i}+ia_1\frac{d}{da_{i+1}}+\frac{i(i+1)}{1\,.\,2}a_2\frac{d}{da_{i+2}}+\&c.,$$

and F_i denote $a_0\dfrac{d}{da_i}+(i+1)a_1\dfrac{d}{da_{i+1}}+\dfrac{(i+1)(i+2)}{1\,.\,2}a_2\dfrac{d}{da_{i+2}}+\&c.$;

(1) express $(F_1)^n$ in terms F_1, F_2, F_3, &c.; and (2) I being an invariant of the i^{th} order of $(a_0, a_1, a_2 \dots a_n)(x, y)^n$, which becomes I′ when every suffix is increased by unity, show that $I = \phi I'$, where

$$\phi = \Sigma \frac{E_r^\lambda \,.\, E_s^\mu \,.\, E_t^\nu \dots}{\lambda\,!\,\mu\,!\,\nu\,!\dots},$$

and $\lambda, \mu, \nu \dots r, s, t \dots$ are any integers satisfying the equation

$$\lambda r + \mu s + \nu t + \dots = i.$$

Solution by W. J. C. Sharp, M.A.

Let $F_1 = F_1' + F_1''$, where F_1 applies only to the function operated upon, and F_1'' only to the a's as involved in the operative symbols. Then

$$F_1''F_1' = 2\,!\,F_2', \quad (F_1'')^2 F_1' = 2F_1''F_2' = 3\,!\,F_3',$$

and generally $\qquad (F_1'')^n F_1' = (n+1)\,!\,F_n';$

therefore $\qquad (F_1)^2 = (F_1' + F_1'')\,F_1 = 2\,!\left(\frac{F_1'^2}{2\,!} + F_2'\right),$

$$(F_1)^3 = (F_1' + F_1'')(F_1)^2 = 3\,!\left(\frac{F_1'^3}{3\,!} + F_1'F_2' + F_3'\right), \text{ \&c. \&c.}$$

and generally $\qquad \frac{(F_1)^n}{n!} = \Sigma \frac{F_r'^\lambda \,.\, F_s'^\mu \,.\, F_t'^\nu \dots}{\lambda\,!\,\mu\,!\,\nu\,!\dots},$

where $\qquad \lambda r + \mu s + \nu t + \dots = n.$

Again, I′ is the same invariant of $v \equiv (a_1, a_2, \dots a_{n+1})(x, y)^n$, that I is of u, and the corresponding invariant of $v + \theta u \equiv (1 + \theta E_1)\,v$,

$$i.e., \left(1 + \frac{\theta E_1}{1\,!} + \frac{\theta^2 E_1^2}{2\,!} + \dots + \frac{\theta^i E_1^i}{i\,!}\right) I', \ \therefore\ I = \frac{E_1^i}{i\,!} I'.$$

But, since I′ is an invariant of v,

$$\left(a_1 \frac{d}{da_2} + 2a_2 \frac{d}{da_3} + \text{\&c.}\right) I' = 0,$$

and consequently $E_1 I' = F_1 I'$ and $E_1^i I' = F_1^i I$ and $E_r^\lambda I' = F_r^\lambda I'$, therefore

$$I = \frac{E_1^i}{i!} I' = \frac{F_1^i}{i!} I' = \Sigma \frac{F_r'^\lambda \,.\, E_s'^\mu \,.\, F_t'^\nu \dots}{\lambda\,!\,\mu\,!\,\nu\,!\dots} = \Sigma \frac{E_r^\lambda \,.\, E_s^\mu \,.\, E_t^\nu \dots}{\lambda\,!\,\mu\,!\,\nu\,!\dots} I',$$

where the E only operate on I′.

[A solution of Quest. 6218 is given in *Reprint*, Vol. 34, p. 99, but it is inaccurate, from the omission of the coefficients in the preliminary equations. A solution of Quest. 7454 is given on p. 113 of Vol. 40 of *Reprints*.]

7664. (By Professor Crofton, F.R.S.)—Prove that the chance of heads turning up twice running during r tosses of a coin is equal to the chance of a run of three (either heads or tails) during $(r+1)$ tosses.

Solutions by (1) E. L. Raymond, M.A.; (2) D. Biddle.

1. Let α be the chance of two heads turning up in the first two tosses; β the chance of two heads turning up in the second and third tosses (and not before); ρ the chance that two heads turn up in the $(r-1)^{\text{th}}$ and r^{th} tosses (and not before); then the chance of turning up two heads running in r tosses is

$$\alpha+\beta+\ldots+\rho \qquad \text{(A)}.$$

Now the chance of turning up a head in the third throw is, of course $\frac{1}{2}$; hence $\frac{1}{2}\alpha$ is the chance of the first three throws being all heads; $\frac{1}{2}\beta$ is the chance of the second, third, and fourth throws being all heads; $\frac{1}{2}\rho$ is the chance of the $(r-1)^{\text{th}}$, r^{th}, and $(r+1)^{\text{th}}$ tosses being all heads; and thus the chance of turning up three heads running in $(r+1)$ tosses is

$$\tfrac{1}{2}(\alpha+\beta+\ldots+\rho) \qquad \text{(B)},$$

and, of course, the chance of three tails being turned up running in $(r+1)$ tosses is

$$\tfrac{1}{2}(\alpha+\beta\ldots+\rho) \qquad \text{(C)}.$$

Hence the chance of a run of three (heads or tails) in $(r+1)$ tosses is

$$\alpha+\beta\ldots+\rho \qquad \text{(D)},$$

which agrees with (A).

2. This proposition postulates two things which themselves require proof, viz., (1) that at every toss of a coin, there is an equal chance as to head or tail turning up, and (2) that all the possible results of r tosses are equally probable. But if *head* have already been turned up twice, it seems only reasonable to assign a greater probability to *tail* for the third, and as the average results must needs be equal for head and tail, in the long run, if a fair coin be fairly tossed, it seems only reasonable to consider the series of events, however short, which approximates to this average, as more probable than one which consists (for instance) of one sort only. The possible ways in which the events may occur double with every increase in the number (r), as may be seen in the adjoining diagram, in which × and 0 represent the only possible results of the first toss; × ×, 0 ×, × 0, and 0 0, of the second; and so on. Consequently, if we consider the probability to be equal as to head or tail at the third toss, no matter what the other two may have yielded, then × × × or 0 0 0 will one or other turn up as the sum of the results once in four times, as × × is supposed to have done when r was 2; and as $\frac{3}{8}$ has been the probability of × × occurring together when r is 3, $\frac{6}{16}=\frac{3}{8}$ is the probability of × × × or 0 0 0 occurring when r is 4. But the matter is open to grave doubt. How do we know that × 0 and 0 × are not twice as probable as × × and 0 0? and × × 0 or 0 0 × four times as probable as × × × or 0 0 0?

4, 3, 2, 1	×	×	×	×
	0	×	×	×
	×	0	×	×
	0	0	×	×
	×	×	0	×
	0	×	0	×
	×	0	0	×
	0	0	0	×
	×	×	×	0
	0	×	×	0
	×	0	×	0
	0	0	×	0
	×	×	0	0
	0	×	0	0
	×	0	0	0
	0	0	0	0

Let us suppose such to be the case. The probability of head turning up in the first toss will then be, as before, $\frac{1}{2}$; but of × × in two, only $\frac{1}{6}$, and of × × × in three, only $\frac{1}{30}$, or of × × × or 0 0 0 (one or other), $\frac{1}{15}$.

Yet in each column there is an equal number of each, *e.g.* in set 3, 15 × + 15 0.

3	x o o o o o o o o x x x x x x o o o o o o x x x x x x x x o x x x x x o o o o o o o o o o x x x x x x x x x x o o o o o x x x x x x x x x x x x x x x o o o o o o o o o o o o o o o
2	x o o x x o x x x o o o
1	x o

What makes us doubt the fairness of a coin which turns up heads (or tails) 100 times in succession, if this be no less probable, in a fair coin fairly tossed, than 50 of each alternately?

7665. (By Professor TOWNSEND, F.R.S.)—The motion of a system of waves, propagated by small rectilinear vibrations in an isotropic elastic solid under the action of its internal elasticity only, being supposed to produce irrotational strain of the substance throughout the entire space and time of vibration; determine, given the coefficients μ and ν of resistance to changes of volume and form of the solid, the differential equation for the potential of the strain at any instant of the motion.

Solution by the PROPOSER.

The strain of the substance being by hypothesis irrotational throughout the entire space and time of vibration, the differential equations of the propagation of the wave system assume in consequence the simplified forms

$$\frac{d^2\xi}{dt^2} = a^2\frac{d\omega}{dx},\quad \frac{d^2\eta}{dt^2} = a^2\frac{d\omega}{dy},\quad \frac{d^2\zeta}{dt^2} = a^2\frac{d\omega}{dz},$$

where $a^2 = \rho^{-1}(\mu + \frac{4}{3}\nu)$ = square of velocity of plane waves propagated by normal vibrations in the substance, and $\omega = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}$ = the cubical dilatation at the point xyz of its mass. But, for irrotational strain of the entire vibrating mass throughout the entire time of vibration, we have also, for all values of x, y, z, and t, within the space and time of vibration, $\xi = \frac{d\phi}{dx}$, $\eta = \frac{d\phi}{dy}$, $\zeta = \frac{d\phi}{dz}$, where ϕ = the required potential of the motion. Therefore the equations of propagation of the wave system, the original coordinates x, y, z of every vibrating molecule being of course independent of the time t of the motion, may be written in the forms

$$\frac{d}{dx}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dx},\quad \frac{d}{dy}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dy},\quad \frac{d}{dz}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dz},$$

from which it follows at once that

$$\frac{d^2\phi}{dt^2} = a^2\omega = a^2\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right);$$

which accordingly is the general equation, in partial differences of the second order, for the potential ϕ of the strain as a function of the coordinates x, y, z of the molecule and the time t of the motion: an equation identical

in form with that for the potential of the propagation of sound in a uniform medium of indefinite extent, and leading of course to the same consequences so familiar in that case.

For plane waves advancing in any common direction in the substance: since then $x\cos\alpha + y\cos\beta + z\cos\gamma = p$, where α, β, γ are the direction angles of propagation and p the distance of any wave plane from the origins, and since consequently $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dp^2}$, therefore the equation for the determination of ϕ assumes the simplified form $\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dp^2}$, the complete integral of which in finite terms, viz., $\phi = k\,.f(p \pm at)$, where k is a small constant representing the absolute amplitude of the vibrations, and f any arbitrary periodic function oscillating within finite limits as in vibratory motion generally, represents two systems of waves advancing in opposite directions with the common velocity a; the amplitudes of vibration undergoing no change for either system with its progress through the substance, and no wave of either system giving rise to a wave of the opposite system in its passage through any portion of the mass external to the region of the original disturbance giving rise to the motion.

For spherical waves diverging from any common centre in the substance: since then $x^2+y^2+z^2 = r^2$, where r is the radius of any wave sphere of the system; and since consequently

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr},$$

therefore the equation for the determination of ϕ assumes again the simplified form $\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right)$, or, in consequence of the entire independence of r and t, the equivalent form $\frac{d^2(r\phi)}{dt^2} = a^2\frac{d^2(r\phi)}{dr^2}$; the complete integral of which in finite terms, viz., $r\phi = k\,.f(r \pm at)$, where k and f are as before, represents again two systems of waves diverging in opposite directions from the common centre with the common velocity a, the amplitudes of vibration varying at considerable distances from the centre inversely as the distance for each, and no wave of either system giving rise to a wave of the opposite system in its passage through any portion of the mass external to the region of the original disturbance giving rise to the motion.

7170. (By the EDITOR.)—If 10 cards are taken at random from a pack, show that the respective probabilities (p_1, p_2) that they will contain (1) exactly 4 cards with hearts, (2) not more than 4 cards with hearts, are

$$p_1 = \frac{27417}{185932} = \cdot 14746 = \frac{5}{34} \text{ nearly}, \quad p_2 = \cdot 94307 = \frac{32}{34} \text{ nearly}.$$

Solution by D. BIDDLE.

1. The following considerations will enable us to build up the required fraction :—$52\,!$ = possible permutations of the whole pack ; $10\,!\ 42\,!$ = ways

of arranging the pack when a particular set of 10 cards is partitioned off and constantly taken first: *e.g.*, 4 hearts and 6 other cards; $\frac{13!}{9!}$ = ways in which 4 cards can be taken out of 13; $\frac{13!}{9!\,4!}$ = different sets of 4 that can be made out of 13, disallowing mere differences of arrangements; $\frac{39!}{33!}$ and $\frac{39!}{33!\,6!}$ = corresponding numbers in regard to 6 cards out of 39. $10!\,42!\,\frac{13!}{9!\,4!}\cdot\frac{39!}{33!\,6!}$ = number of ways in which 4 cards of one kind and 6 cards of other kinds can be arranged on one side of a partition, and the remainder of the pack on the other side, by allowing interchange of equivalent cards across the partition, and permutation on each side. Hence the probability that 10 cards drawn at random from a pack will contain exactly 4 hearts is $\frac{10!\,42!\,13!\,39!}{52!\,9!\,4!\,33!\,6!}$. This, reduced, is ·14746 nearly, or rather over $\frac{5}{34}$. Consequently, a person accepting odds of 6 to 1, laid against the occurrence, should win in the long run.

2. The probability that the 10 cards contain not more than 4 hearts is the sum of the respective probabilities concerning 4, 3, 2, 1, 0 which (as found by the formula in Art. 1) are as follows:—4 hearts = ·14746; 3 h. = ·27807; 2 h. = ·30334; 1 h. = ·17404; and 0 h. = ·04016. But the sum may be stated in the following manner:—

$$\frac{10!\,42!\,13!\,39!}{52!}\left(\frac{1}{9!\,4!\,33!\,6!}+\frac{1}{10!\,3!\,32!\,7!}+\frac{1}{11!\,2!\,31!\,8!}\right.$$
$$\left.+\frac{1}{12!\,1!\,30!\,9!}+\frac{1}{13!\,29!\,10!}\right).$$

Hence *not more than* 4 *h.* = ·94307. This is slightly over $\frac{32}{34}$.

[Putting $C_n^{(r)}$ for the number of combinations of n things taken r together, the total number of ways and the number of favourable cases in (1) are respectively $C_{52}^{(10)}$, $C_{13}^{(4)}\times C_{39}^{(6)}$ (since each group of 4 hearts out of 13 may be combined with each group of 6 non-hearts out of 39); hence the probability in question is $C_{13}^{(4)}\times C_{39}^{(6)} \div C_{52}^{(10)}$, which gives the first result; and the sum of 5 such probabilities gives the second result.]

6456. (By G. Heppel, M.A.) — If the expansion of sec x be $1+\frac{u_2}{2!}x^2+\frac{u_4}{4!}x^4+$ &c., and that of $2\sec^2 x$ be $2+\frac{v_2}{2!}x^2+\frac{v_4}{4!}x^4+$ &c.; prove that the coefficients u_2, u_4, u_6, &c. may be found from the relations

$$u_{2n}=u_{2n-2}+\frac{2n-2!}{2!\,2n-4!}\,u_{2n-4}\cdot v_2+\frac{2n-2!}{4!\,2n-6!}\,u_{2n-6}\cdot v_4+\text{\&c.},$$

and $$\tfrac{1}{2}v_{2n}=u_{2n}+\frac{2n!}{2!\,2n-2!}\,u_2\,u_{2n-2}+\frac{2n!}{4!\,2n-4!}\,u_4\,u_{2n-4}+\text{\&c.},$$

taking care in the last series to stop before the first suffix exceeds the second, and to halve the coefficient when they become equal. Apply this

method to verify the value of u_{10}, found (by another process) in DE MORGAN'S *Differential Calculus*, to be 50521.

Solution by the Rev. T. C. SIMMONS, M.A.

Differentiating twice with respect to x the identity

$$\sec x \equiv 1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \dots,$$

we have

$$\frac{u_{2n}}{2n-2!} = \text{coefficient of } x^{2n-2} \text{ in the expansion of } \sec x \, (2 \sec^2 x - 1)$$

$$= \text{same coefficient in } \left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \dots\right)\left(1 + \frac{v_2 x^2}{2!} + \frac{v^4 x^4}{4!} + \dots\right)$$

$$= \frac{u_{2n-2}}{2n-2!} + \frac{u_{2n-4}}{2n-4!} \cdot \frac{v_2}{2!} + \frac{u_{2n-6}}{2n-6!} \cdot \frac{v_4}{4!} + \dots,$$

whence follows the first required relation.

Again, $$\frac{1}{4}\,\frac{v_{2n}}{2n!} = \text{coefficient of } x^{2n} \text{ in } \tfrac{1}{2} \sec^2 x$$

$$= \text{same coefficient in } \frac{1}{2}\left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \dots\right)\left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \dots\right)$$

$$= \frac{u_{2n}}{2n!} + \frac{u_{2n-2}}{2n-2!} \cdot \frac{u_2}{2!} + \frac{u_{2n-4}}{2n-4!} \cdot \frac{u_4}{4!} + \dots,$$

with the reservation stated in the question.

Now put n successively equal to 2, 3, 4, 5; then, observing that u_0 and n_2 each $= 1$, we find $u_4 = 5$, $v_4 = 32$, $u_6 = 61$, $v_6 = 544$,

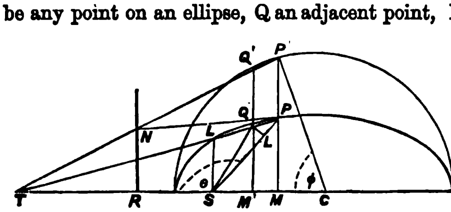
$$u_8 = 61 + 300 + 480 + 544 = 1385, \quad v_8 = 4\,(1385 + 1708 + 875) = 15872,$$

$$u_{10} = 1385 + 6832 + 11200 + 15232 + 15872 = 50521.$$

7733. (By H. J. READ, M.A.) — Transform $\int \frac{d\theta}{(1 + e\cos\theta)^{n+1}}$, by means of the geometry of the ellipse.

Solutions by (1) D. EDWARDES; (2) H. L. ORCHARD, B.Sc., M.A.

1. Let P be any point on an ellipse, Q an adjacent point, P′, Q′ corres-



ponding points on the auxiliary circle; then, drawing QL perpendicular to SP, $$r = \mathrm{SP} = a - ex = a(1 - e\cos\phi).$$
But $r = a(1-e^2)\,1\mathrm{H}e\cos\theta$; $\therefore\ a(1-e^2) = a(1-e\cos\phi)(1-e\cos\theta)\ldots(\alpha)$.
Also $$\mathrm{MM}' = \mathrm{PQ}\cos\mathrm{PTM} = \mathrm{P'Q'}\cos\mathrm{P'TM};$$
therefore $$\frac{\mathrm{PQ}}{\mathrm{P'Q'}} = \frac{\mathrm{TQ}}{\mathrm{T'Q'}} = \frac{b'}{a},$$
where b' is the semi-diameter parallel to TQ. But
$$\mathrm{QL} = r\,d\theta = \mathrm{QP}\sin\mathrm{QPL} = \mathrm{PQ}\,\frac{b}{b'},$$
and $a\,d\phi = \mathrm{P'Q'}$, therefore $\dfrac{\mathrm{PQ}}{\mathrm{P'Q'}} = \dfrac{r\,d\theta}{a\,d\phi}\,\dfrac{b'}{b} = \dfrac{b'}{a}$, or $r\,d\theta = b\,d\phi\ldots(\beta)$;
therefore $\displaystyle\int\frac{d\theta}{(1+e\cos\theta)^{n+1}} = \frac{1}{(1-e^2)^{n+\frac12}}\int d\phi\,(1-e\cos\phi)^n$ from (α) and (β).

[The formula (β) seems to be useful; *e.g.*, area of ellipse
$$= 2\,.\,\tfrac12\int_0^\pi r^2\,d\theta = \int_0^\pi r\,.\,r\,d\theta = ab\int_0^\pi(1-e\cos\phi)\,d\phi = \pi ab.]$$

2. *Otherwise:* by the geometry,
$$\mathrm{SP} = e\mathrm{PN} = e\mathrm{MS} + e\mathrm{SR},\quad i.e.,\ r = -er\cos\theta + \mathrm{LS} = -er\cos\theta + l,$$
or, $$1 + e\cos\theta = \frac{l}{r};$$
hence $$\frac{l\,dr}{r^2} = e\sin\theta\,d\theta = \frac{[r^2(e^2-1) + 2lr - l^2]^{\frac12}}{r}\,d\theta$$
$$= \left(\frac{2lar - lr^2 - l^2a}{a}\right)^{\frac12}\frac{d\theta}{r}\quad\left(\text{since } 1 - e^2 = \frac{l}{a}\right);$$
whence
$$\int\frac{d\theta}{(1+e\cos\theta)^{n+1}} = \left(\frac{a}{l^{2n+1}}\right)^{\frac12}\int\frac{r^n\,dr}{(2ar - r^2 - al)^{\frac12}} = \frac{a^{n+1}}{b^{2n+1}}\int\frac{r^n\,dr}{(2ar - r^2 - b^2)^{\frac12}},$$
a and b being the semi-axes.

5635. (By Elizabeth Blackwood.)—Two excursion trains, each m yards in length, may start with equal probability from their respective stations at any time between 2 o'clock and 10 minutes past 2, in directions at right angles to each other, each at a uniform rate v; find the chances of a collision, each being n yards distant from the point at which their lines cross, and both being ignorant of the risk they are running.

Solution by D. Biddle.

If v = velocity of each train in miles per hour, then $\frac16 v$ = distance in miles covered in 10 minutes; and, if m = length of train in yards, $\frac{1}{1760}m$ = length referred to a mile as unit. Also $\dfrac{6m}{1760v}$ = fraction of

10 minutes occupied in clearing the level crossing. $10-\dfrac{2.6m}{1760v}$ = time during which a collision can occur when train B starts as much as $\dfrac{6m}{1760v}$ before or after train A. $\dfrac{2.6m}{1760v}$ = time during which the chance is curtailed owing to limitation in the time of starting of the two trains, and when the average factor is reduced from 2 to $\frac{1}{2}(2+1)$. Consequently,

$$\frac{1}{10}\left\{\left(10-\frac{2.6m}{1760v}\right)\frac{2.6m}{1760v}+\frac{2.6m}{1760v}\cdot\frac{3.6m}{2.1760v}\right\} = \text{chance required}$$

$$= \frac{1}{10}\left(20-\frac{6m}{1760v}\right)\frac{6m}{1760v}.$$

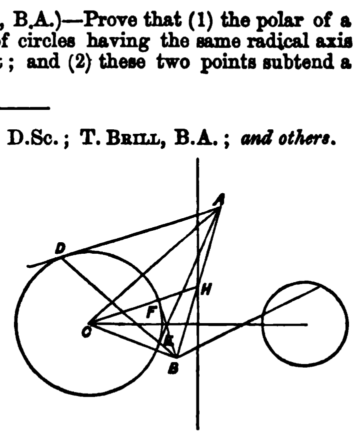
Thus, let $m = 200$, and $v = 35$, then the chance of a collision

$= \frac{1}{10}\left(20-\frac{3}{154}\right)\frac{3}{154} = \cdot 03891$, or rather more than $\frac{10}{257}$.

The distance n of each train from the level crossing does not appear to affect the probability of a collision, under the other specified conditions. And the moral to be drawn is, that in such cases, the higher the speed and the shorter the train, the less the chance of disaster.

7730. (By W. J. Greenstreet, B.A.)—Prove that (1) the polar of a fixed point with regard to a series of circles having the same radical axis passes through another fixed point; and (2) these two points subtend a right angle at either limiting point.

Solution by A. H. Curtis, LL.D., D.Sc.; T. Brill, B.A.; *and others*.



Lemma. The line joining a point A to any point B, situated on the polar of A with respect to any circle, (1) is equal to the sum of the tangents to the circles from A and B, and (2) is equal to the double of the tangent drawn from its middle point H.

Let C be the centre of the circle and a its radius.

$$(1)\ AB^2 = AD^2 + DB \cdot BE = AD^2 + BF^2.$$

$$(2)\ 4AH^2 = AB^2 = AD^2 + BF^2 = AC^2 + BC^2 - 2a^2 = 2CH^2 + 2AH^2 - 2a^2,$$

therefore $AH^2 = CH^2 - a^2$ = square of tangent from H, therefore AB = the double of such tangent.

Let now the polars of A taken with respect to any two circles meet in B, bisect AB in H, then the tangents from H to the two circles are each = AH, therefore equal to each other; therefore H is on the radical axis of the circles, and, as AH = BH, A and B are equidistant from it; if we find then on DE a point B, whose distance from the radical axis = the distance

of A from same, the point so formed is a point through which the polar of A taken with respect to any circle of the system must pass. Again, as AH = BH = tangent from H to any circle of the system = distance of H from either limiting point, the circle on AB as diameter will pass through the limiting points; therefore AB will subtend a right angle at either of these points.

6859. (By Professor SIMON NEWCOMB, M.A.)—Prove that

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\ldots-2\eta\cos x$$

$$-\tfrac{1}{2}\,.\,2\eta^2\cos 2x-\tfrac{1}{3}\,.\,2\eta^3\cos 3x-\ldots = \Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

Solution by ASŪTOSH MUKHOPĀDHYĀY.

By elementary trigonometry,

$$\frac{\eta\sin x}{1-2\eta\cos x+\eta^2} = \eta\sin x+\eta^2\sin 2x+\ldots,$$

therefore $\log(1-2\eta\cos x+\eta^2) = \int\frac{2\eta\sin x\,dx}{1-2\eta\cos x+\eta^2}$

$$= -2\left(\eta\cos x+\tfrac{1}{2}\eta^2\cos 2x+\tfrac{1}{3}\eta^3\cos 3x+\ldots\right).$$

Hence, since $\log(1+\eta^2) = \eta^2-\tfrac{1}{2}\eta^4+\tfrac{1}{3}\eta^6-\ldots$, we have

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = \left(-\eta^2+\tfrac{1}{2}\eta^4+\tfrac{1}{3}\eta^6+\ldots\right)-2\left(\eta\cos x+\tfrac{1}{2}\eta^2\cos 2x+\ldots\right)$$

$$= \Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

[For another solution, see *Reprint*, Vol. 36, p. 116.]

7673. (By Professor COCHEZ.)—La série

$$\frac{1}{2\log 2\,(\log\log 2)^a}+\frac{1}{3\log 3\,(\log\log 3)^a}+\ldots\frac{1}{n\log n\,(\log\log n)^a}$$

est convergente si $a>1$, divergente si $a<1$.

Solution by B. H. RAU, M.A.; BELLE EASTON; *and others*.

Let $\phi(x) = \dfrac{1}{x\log x\,(\log\log x)^a}$, then $\int\phi(x)\,dx = \dfrac{(\log\log x)^{1-a}}{1-a}$,

if a be not unity, and $=$ (log log log x) if p be unity; hence

$$\int_2^\infty\phi(x)\,dx = -\frac{(\log\log 2)^{1-a}}{1-a},$$

if p be greater than unity, and is infinite if p be equal to unity or less than unity. But the definite integral and the given series are both finite or both infinite. Hence the series is convergent when $a > 1$ and divergent when $a < 1$.

7720. (By R. LACHLAN, B.A.)—Four circles, having their centres within the triangle ABC, are drawn to cut the side BC in angles α, 2σ, γ, β; the side CA in angles β, γ, 2σ, α; and the side AB in angles γ, α, β, 2σ respectively: where $2\sigma = \alpha+\beta+\gamma$. Let R_1, R_2, R_3, R_4 be their radii, and r the radius of the inscribed circle; also let these five circles cut any straight line in angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ; then prove that

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} = 4\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\cdot\frac{1}{r},$$

and $\cos\phi_1+\cos\phi_2+\cos\phi_3+\cos\phi_4 = 4\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}\cos\dfrac{\alpha+\beta}{2}\cdot\cos\theta.$

Solution by B. H. RAU, M.A.; Rev. T. C. SIMMONS, M.A.; *and others.*

Let x, y, z be the trilinear coordinates of the centre of the circle (R_1) which cuts the sides BC, CA, AB in angles α, β, γ; then $x = R_1\cos\alpha$, $y = R_1\cos\beta$, $z = R_1\cos\gamma$, $2\Delta = ax+by+cz$,

therefore $$R_1 = \frac{2\Delta}{a\cos\alpha+b\cos\beta+c\cos\gamma}.$$

Similarly $$R_2 = \frac{2\Delta}{a\cos 2\sigma+b\cos\gamma+c\cos\alpha}, \text{ and so on;}$$

therefore $$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} = \frac{a+b+c}{2\Delta}(\cos\alpha+\cos\beta+\cos\gamma+\cos 2\sigma).$$

$$= \frac{4}{r}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}.$$

If $lx+my+nz = 0$ be the equation to any straight line, then

$$\frac{R_1\cos\phi_1}{r\cos\theta} = \frac{\text{perp. from centre of } R_1}{\text{perp. from centre of } r} = \frac{lR_1\cos\alpha+mR_1\cos\beta+nR_1\cos\gamma}{lr+mr+nr};$$

therefore $$\frac{\cos\phi_1}{\cos\theta} = \frac{l\cos\alpha+m\cos\beta+n\cos\gamma}{l+m+n}.$$

Similarly $$\frac{\cos\phi_2}{\cos\theta} = \frac{l\cos 2\sigma+m\cos\gamma+n\cos\alpha}{l+m+n}, \text{ and so on,}$$

therefore $$\Sigma\frac{\cos\phi}{\cos\theta} = \cos\alpha+\cos\beta+\cos\gamma+\cos 2\sigma,$$

therefore $$\Sigma\cos\phi = 4\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)\cos\tfrac{1}{2}(\alpha+\beta)\cdot\cos\theta.$$

7646. (By W. G. LAX, B.A.)—If ABCD be a rectangle; E, G two points in AB, AD such that EOF, GOH meet each other and the diagonal

BD in O, and are parallel respectively to AD, AB; if also AB is taken to represent the external E.M.F of an electro-motor supplied with a current at constant E.M.F; EB that of the motor at a given speed, and the ratio BA : AD the resistance of the circuit: show from this figure what are the conditions for (1) the maximum efficiency, (2) the maximum rate of working; and (3) find expressions for the electrical energy wasted, and that used in work, each per unit of time.

Note by the EDITOR.

Mr. BUCK informs us that the figure is drawn and fully discussed as to all points in the question in Professor S. P. THOMPSON's *Cantor Lectures*, delivered in December, 1882.

The PROPOSER remarks that the theorems were suggested to him by a question he happened to see in an examination paper, and that he had not the slightest idea that they had previously been investigated.

7314. (By G. HEPPEL, M.A.)—One side of a railway carriage is covered with a glass mirror. Show that, while the train is going round a level curve, the images of reflected objects appear to an observer sitting opposite to describe hyperbolas on the glass. Supposing the centre of the railway curve to be 1000 yards from the observer's eye and 900 yards from the object, the distance of the eye from the mirror to be 2 yards, and the height of the object above the eye 10 yards; find the position and magnitude of the axes of the hyperbola. Also discuss the path of the image of a star.

Solution by the PROPOSER.

Let horizontal and vertical axes be taken on the glass through the image of the observer's eye, which is a fixed point. Let the distance of this image from the eye be c, and from the centre of the railway curve be r. Let l be the distance measured horizontally between the object and the centre, and, taking the line joining these as an initial line, let θ be the angular distance travelled by the carriage. Let h be the height of the object above the horizontal plane through the eye. Then, by the first principles of reflection, we shall have

$$\frac{x}{c} = \frac{r - l\cos\theta}{l\sin\theta + c}; \quad \frac{y}{h} = \frac{c}{l\sin\theta + c};$$

therefore $$l\sin\theta = \frac{c(h-y)}{y}; \quad l\cos\theta = r - \frac{hx}{y};$$

therefore $$\frac{c^2(h-y^2)}{y^2} + \left(r - \frac{hx}{y}\right)^2 = l;$$

$$(r^2+c^2-l^2)y^2 - 2rhxy + h^2x^2 - 2c^2hy + c^2h^2 = 0;$$

and this, except in the very exceptional case of l being less than c, is a hyperbola.

Taking the numbers given in the example, neglecting c^2 in comparison with l^2 and r^2, and expressing all lengths in inches, we find that the centre of the hyperbola is $x = -1\cdot\dot{7}$, $y = -\cdot 001\dot{7}$; the inclination of the trans-

verse axis is 3° 0′ 22″; and the semi-axes are 34·926 and 1·77. In the case of a star, if α be the altitude, l and h are both infinite, and $\frac{l}{h} = \tan\alpha$; the hyperbola then becomes $y^2 = (x^2+c^2)\tan^2\alpha$.

[Mr. BIDDLE remarks that "there are two arrangements by which the specified conditions can be met. The observer can be sitting with his back to the centre of the curve, the mirror being on the other side of the (saloon) carriage and tangential to the curve (A); or he may be facing a mirror whose plane contains the centre of the curve (B).

"The object, being nearer to the centre of the curve than the carriage, must lie within the circle of which the railway curve forms part; and it is immaterial whether the circular motion be effected by the *carriage* (with observer and mirror inside) or by the *object*, provided the centre of revolution be the same: the path of the image on the glass will be identical. Moreover, a circular ring representing the object in all possible positions at the same moment of time will, produce an image representing the path required, and this image will be identical with that which would be produced if the observer were looking through the glass at a similar circle on the other side; and, if he were to trace the outline in the manner recommended by Mr. RUSKIN (in his treatise on Perspective), he would find the curve to be ***elliptical***, **NOT** ***an hyperbola***.

"In fact, anyone who will take the trouble to draw a circle on the ceiling of his drawing-room, and look at it as reflected in the pier-glass (whether in the position A or B), will have an exemplification in miniature of the same thing.

"The path of the image of a star differs but slightly in principle from the foregoing. The star being outside the circle and at an infinite distance, so to speak, as compared with the diameter of the railway curve, would produce an image crossing the glass almost in a straight line. In reality, it would be a portion of an ellipse with comparatively small minor axis. In position (B) the mirror cannot reflect more than half any ring concentric with the railway curve at any one time."

[The PROPOSER "regrets that his question was not so clearly worded as to show without doubt that 'side' meant the partition against which the seats are placed, and not where the doors are. Adopting the method and illustration in the remarks, it may be stated that the curve on the glass is certainly an ellipse in case A, but this was not the case meant. In case B the effect would be the same as if an observer looked in the glass at the reflection of a circle on the ceiling, the centre of which was in the line of intersection of the wall and ceiling. If a cone be imagined, defined by such a circle and by the observer's eye as vertex, it is clear that, if the distance of the eye from the glass were less than the radius of the circle, the wall would cut the cone on both sides of the vertex, and the section would be a hyperbola; if equal, a parabola; and if greater, an ellipse. Returning from the illustration to the thing illustrated, it will be found that this is equivalent to considering whether the distance of the eye from the glass is less or greater than that of the object horizontally from the centre of the curve; and practically it is always less."]

7685. (By Rev. H. E. DAY, M.A.)—Find the probability of a piece at Chess being found on any particular square after having been moved at

random an indefinitely long time. [By a well-known mathematical law, this limit is only strictly true after an *indefinitely* long time, but approximately true after a considerable number of moves. For the case of the Knight, see the solution of Quest. 4955, in *Reprint*, Vol. 25, p. 78.]

Solution by D. BIDDLE *and the* PROPOSER.

Let A be the chance of its being found on one particular square (with a outlets) communicating with squares having b, c, d, e, f outlets respectively, and whose several chances in regard to the piece are in like manner B, C, D, E, F.

Then $A = \frac{B}{b} + \frac{C}{c} + \frac{D}{d} + \text{\&c.}$ to a terms, since the square with chance A is one of the outlets of the squares with chance B, C, &c., and shares in due proportion their several chances, each outlet taking an equal share of the chance pertaining to the square of which it is an outlet. Equations for B, C, &c. can be obtained in like manner; and the several results arrived at, that is, the values of A, B, C, &c., though not identical, are constant; each is a fixed and invariable amount. But evidently any such equation is satisfied by $\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d} = \frac{E}{e}$ because all the squares on the board are outlets, proximate or remote, of one another, and the several equations carried out and reduced would give no other result. Such being the case,

$$\frac{A}{a} = \frac{A+B+C+\text{\&c.}}{a+b+c+\text{\&c.}} = \frac{\text{the sum of the chances for the whole board}}{\text{the sum of the outlets for the whole board}}$$

$$= \frac{1}{a+b+c+\text{\&c.}};$$

Consequently $A = \frac{a}{a+b+c+\text{\&c.}}$, and is proportional to the outlets (or which is the same thing, the *inlets*) of the particular square, so far as the particular piece is concerned.

7430. (By Professor HUDSON, M.A.)—Given three points, determine in how many ways they may be the positions of an eye, a luminous point, and its image formed by reflexion at a plane mirror; and construct in each case the position of the mirror.

Solutions by (1) W. J. C. SHARP, M.A.; (2) D. BIDDLE.

1. Let A, B, C be the three points; then, if the eye be supposed to be at A, the mirror must be the perpendicular bisector of BC, and the object must be on the same side of this as A; hence there will be one arrangement with the eye at A if AB and AC are unequal, and none if these are equal. So for each of the other points; and the eye may be placed at each, at two, or at none of the points, according as the triangle is scalene, isosceles, or equilateral; and, whenever the eye occupies any angle, the mirror is the perpendicular bisector of the opposite side of the triangle formed by the points.

2. The optical requirements, in the above series of cases, are (1) that the eye and the object shall invariably be in front of the mirror, and the image behind it; and (2) that the reflecting surface, whilst placed directly between the eye and the image, shall occupy a plane which bisects at right angles the line joining the image and the object. This secures the needed equality of the angles of incidence and reflexion, and also respects the apparent equality which subsists between the perpendicular distance of the object in front and of the image behind from the plane of the mirror.

If the lines joining the three points form an equilateral triangle, it is impossible that the desired optical effect can be produced, unless the position of the eye be allowed to coincide with the plane of the mirror; and in no other case can the entire series of permutations, six in number, be accomplished. If the lines joining the three points form an isosceles triangle (not equilateral), there are two arrangements possible: for, if the angle at the apex be less than 60°, the points at the base may be those indifferently of the eye or the object; and, if the angle be greater than 60°, the points at the base may be those indifferently of the eye or the image. When the lines joining the three points form a scalene triangle, three arrangements are possible, the eye taking each of the three positions in turn, and the image the more distant of the two remaining. A diagram is scarcely needed to make this clear.

6553. (By the late G. F. WALKER, M.A.)—Solve the equations
$$x^2(y+z) = a^3, \quad y^2(z+x) = b^3, \quad z^2(x+y) = c^3.$$

Solution by the Rev. T. C. SIMMONS, M.A.

[As the final equations in the solution of this question given on p. 86 of Vol. xxxv. of our *Reprints*, are evidently *not* three ordinary equations for finding x, y, z, but only their *ratios*, the following method is suggested.]

Putting $xyz = -p$, the equations are
$$\frac{a^3}{px} + \frac{1}{y} + \frac{1}{z} = 0, \text{ \&c.} \quad\ldots\ldots\ldots\ldots\ldots(1), (2), (3),$$

whence $\begin{vmatrix} \frac{a^3}{p} & 1 & 1 \\ 1 & \frac{b^3}{p} & 1 \\ 1 & 1 & \frac{c^3}{p} \end{vmatrix} = 0$, or $2p^3 - (a^3+b^3+c^3)p^2 + a^3b^3c^3 = 0$. Let any root of this equation be $-k$, then, solving (1), (2), (3), we obtain $x = mz$, $y = nz$, hence $mnz^3 = k$, therefore z, and consequently x and y, are determined.

[This method at once solves the more general equation $x^2(y+\lambda z) = a^3$, $y^2(z+\mu x) = b^3$, $z^2(x+\nu y) = c^3$, or, what is the same thing, the equations $x^2(ay+bz) = y^2(cz+dx) = z^2(ex+fy) = 1$, which can only be solved with great difficulty by the ordinary methods. A similar remark applies to Mr. SIMMONS's solution of Quest. 7341 (p. 29 of this volume), in the form
$$yz(ax+by+cz) = d, \quad zx(a'x+b'y+c'z) = d', \quad xy(a''x+b''y+c''z) = d'',$$

which possibly cannot be solved *at all* by the ordinary method. By this new determinant process a whole host of awkward-looking equations can be solved *at once;* whereof two specimens are hereunder :—

$$x^2(1+y^2) = ayz, \quad y^2(1+z^2) = bzx, \quad z^2(1+x^2) = cxy,$$
$$ax+by = x^2y^2, \quad cy+dz = y^2z^2, \quad ez+fx = z^2x^2.]$$

7700. (By W. J. McClelland, B.A.)—Prove that

$$\left(\frac{1+\cos a-\cos b-\cos c}{\sin\frac{1}{2}b\sin\frac{1}{2}c}\right)^2+(\ldots)^2+\left(\frac{1+\cos c-\cos a-\cos b}{\sin\frac{1}{2}a\sin\frac{1}{2}b}\right)^2$$
$$+\frac{(1+\cos a-\cos b-\cos c)\ldots(1+\cos c-\cos a-\cos b)}{2\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b\sin^2\frac{1}{2}c} = 16.$$

Solution by B. H. Rau, M.A.; T. Trap, B.A.; *and others.*

Let $x, y, z \equiv 1-\cos a,\ 1-\cos b,\ 1-\cos c$, and $u \equiv x+y+z$; then

$$\text{Sinister} = \frac{4(u-2x)^2}{yz}+\frac{4(u-2y)^2}{zx}+\frac{4(u-2z)^2}{xy}+\frac{4(u-2x)(u-2y)(u-2z)}{xyz}$$
$$= \frac{4}{xyz}\left\{u^2(x+y+z)-4u(x^2+y^2+z^2)+4(x^3+y^3+z^3)\right.$$
$$\left.+u^3-2u^2(x+y+z)+4u(xy+yz+zx)-8xyz\right\}$$
$$= \frac{16}{xyz}\left\{x^3+y^3+z^3-2xyz-u(x^2+y^2+z^2-yz-zx-xy)\right\} = 16.$$

[*Otherwise.*—Suppose a, b, c to represent the arcs of a spherical triangle, then $\dfrac{1+\cos a-\cos b-\cos c}{4\sin\frac{1}{2}b\sin\frac{1}{2}c} = \cos A'$, where A′ is the angle of the corresponding chord at triangle, and since

$$A'+B'+C' = \pi, \quad \cos^2 A'+\cos^2 B'+\cos^2 C'+2\cos A'\cos B'\cos C' = 1.]$$

7722. (By Rev. H. G. Day, M.A.)—If there be m black squares, and n white ones, find the chance that $\alpha+\beta$ pieces placed at random will cover α black and β white.

Solution by the Proposer.

Let $(p)_q$ denote the number of combinations of p things q together; then the total number of ways in which the $\alpha+\beta$ pieces can be placed is $(m+n)_{\alpha+\beta}$. But the ways in which α black and β white squares can be covered is $(m)_\alpha(n)_\beta$; hence the required probability is $\dfrac{(m)_\alpha(n)_\beta}{(m+n)_{\alpha+\beta}}$.

For the chance of there being β obstacles between a bishop and king at opposite diagonal corners, we must take $m = 56$, $n = 6$, $\alpha+\beta = 4$.

7583. (By MORGAN JENKINS, M.A.)—Prove GERGONNE's construction for describing a circle to touch three given circles without introducing, in the proof, two tangent circles.

Solution by the PROPOSER.

Let A, B, C be the centres of the three given circles, O of the orthogonal circle, T of one of the required circles; and let the circles, for brevity, be named by their centres. Let h, k be the points of contact of the circle T with the circles A and B respectively; $O\alpha$, $O\alpha'$ tangents from O to the circle A; $O\beta$, $O\beta'$ to the circle B. Let $\alpha\alpha'$ meet the common tangent at h to the circles A and T in H; and $\beta\beta'$ meet the common tangent at k in K, and Hh, Kk meet in t. Then, because Hh is the radical axis of the circles T and A, and $H\alpha\alpha'$ of the circles O and A, therefore H is on the radical axis of the circles O and T. Similarly K is on the same radical axis. Therefore HK is the radical axis of the circles O and T, and is perpendicular to the line OT. Again, since th and tk are tangents to the circles A and B, hk passes through a centre of similitude of these two circles. For the same reason, $\alpha\beta$ and $\alpha'\beta'$ pass through one centre of similitude of the circles A and B, $\alpha\beta'$ and $\alpha'\beta$ through the other centre of similitude. Of the two centres of similitude, one, say the former, must be that through which hk passes; let it be denoted by S: then, since rect. $S\alpha \,.\, S\beta =$ rect. $Sh \,.\, Sk$, S is on the radical axis of the circles O and T, that is, HK passes through S. Similarly, if γ, γ', l, L be points corresponding, for the circle C, to the points α, α', h, H, then HL is perpendicular to OT, that is, coincident with HK; and HKL passes through a centre of similitude of the circles A and C, and through a centre of similitude of the circles A and B. Therefore the radical axis of the circles O and T is an axis of similitude of the circles A, B, and C.

Again, since h is the pole of Hh, O of $\alpha\alpha'$ with regard to the circle A, therefore Oh is the polar of H, therefore the pole of HKL lies in Oh, therefore h may be determined by joining O to the pole, with regard to the circle A, of an axis of similitude of the three circles.

[If the three circles A, B, C cut one another two and two, the point O, the intersection of common chords, is inside each of the circles; the orthogonal circle and the six points α, α', β, β', γ, γ' are imaginary; the lines $\alpha\alpha'$, $\beta\beta'$, and $\gamma\gamma'$ are the polars of O with regard to the three circles respectively. If the circles A and B cut each other in q, q', and if SAB cut qq' in n,

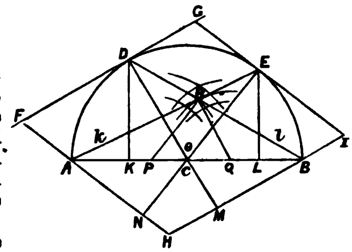
$$\text{rect. } Sh\,.\,Sk = Sq^2 = Sn^2 + nq^2 = Sn^2 + On^2 + Oq\,.\,Oq'$$
$$= SO^2 + \text{rect. } Oq\,.\,Oq' = SO^2 - \text{square of radius of the}$$

imaginary circle, with centre O, cutting the three circles orthogonally. Therefore, as before, S is on the radical axis of the circles O and T, and the remaining part of the proof holds good.]

5672. (By Col. CLARKE, C.B., F.R.S.)—P and Q are two points in a finite line AB. The parts PA, QB are rotated in opposite directions round P and Q respectively, until A and B meet in a point R. Supposing P and Q evenly distributed, determine the law of density of the points R.

Solutions by (1) D. BIDDLE; (2) *the* PROPOSER.

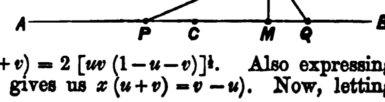
1. In order to meet in a point R, PA and QB must bear a certain relation to each other; that is, Q must be within the same distance of C (the mid-point of AB) that P is of A; it must also be on the opposite side of C. Again, if we divide AC and CB into a number of equal parts, and draw a series of semicircles from each extremity of AB, with the diameters in arithmetical progression, it is easy to see that the spaces enclosed by arcs, though of different size and shape, will each contain the same number of possible positions for R, because the distance between any two adjoining centres on AB is the same, affording scope for equal numbers of P's and Q's.



Now, since the distance between any two semicircles drawn from one extremity of AB increases as they proceed in the same proportion as chords of the full-sized semicircle drawn on AB itself, we have a ready means of estimating for any position the proportionate intervals between adjacent semicircles in both series. They correspond with the chords which pass through the point in question, from A and B respectively, to the semicircle described on AB.

In order to estimate the relative extent of space occupied by R in its several positions, we must take a space bounded by arcs so minute as to be virtually straight lines, as in the diagram, where from P and Q a few concentric arcs are described and small portions of them magnified to form the quadrilateral FGIH. C being the centre of the semicircle, CD, CE are parallel to QR, PR; GI, FH are at right angles with the direction of PR, and FG, HI likewise with that of QR. Moreover GI, FH are separated by the chord AE, and FG, HI by the chord BD, lying in their normal positions. But the perpendicular distance between FG and HI = BK, and between GI and FH = AL; and these (AB being unity) are respectively BD^2 and AE^2. Also the angles F and I are each = DCE = PRQ. The area of FGIH = HI . DM = HI . BK = AL cosec I . BK = $AE^2 . BD^2$ cosec DCE. And the same rule applies to every position of R. If therefore θ = angle separating radii to upper extremities of the two chords k, l; then the extent of space occupied by R at the intersection of the chords may be represented by $k^2l^2 \operatorname{cosec}\theta$; and the density of points R in the same locality by $1/k^2l^2 \operatorname{cosec}\theta$. We thus see that the density on the circumference and also on the base line is infinitesimal compared with the density just within the corners of the semicircle.

2. *Otherwise:* — Let half AB, that is AC, be taken as the unit of length. Let CM = x, MR = y. Also, let PC = u, CQ = v, then the double area of PQR is $y(u+v) = 2[uv(1-u-v)]^{\frac{1}{2}}$. Also expressing the difference of PM and QM gives us $x(u+v) = v-u$. Now, letting



u and v receive increments du and dv, R becomes one of the four corners of a small parallelogram whose coordinates are

$$\begin{array}{ll} x & y \\ x+\dfrac{dx}{du}\,du & y+\dfrac{dy}{du}\,du \\ x+\dfrac{dx}{dv}\,dv & y+\dfrac{dy}{dv}\,dv \\ x+\dfrac{dx}{du}\,du+\dfrac{dx}{dv}\,dv & y+\dfrac{dy}{du}\,du+\dfrac{dy}{dv}\,dv, \end{array}$$

the area of which is $\left(\dfrac{dx}{dv}\dfrac{dy}{du}-\dfrac{dx}{du}\dfrac{dy}{dv}\right)du\,dv,$

and the reciprocal of this is a measure of the density of distribution of the points. We find readily $\dfrac{dx}{dv}\dfrac{dy}{du}-\dfrac{dx}{du}\dfrac{dy}{dv}=\dfrac{y}{u+v}\cdot\dfrac{1}{1-u-v}.$

But, since $\quad 2u=1-x-\dfrac{y^2}{1+x}$ and $2v=1+x-\dfrac{y^2}{1-x}$,

the expression for the density becomes $\dfrac{y\,(1-x^2-y^2)}{(1-x^2)^2}.$

7572. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In the limaçon whose equation is $r=a\cos\theta+b$, where $b>a$, O is the origin, A, A′ the farther and nearer vertices, C a point of maximum curvature, P, P′ two points of the curve on the same side of the axis as C, such that OC is the harmonic mean between OP, OP′ (P coinciding with A when P′ coincides with A′); prove that (1) the difference of the angles AOP, A′OP′ is equal to the angle (ϕ) which the chord PP′ makes with the axis; the difference of the arcs AP, A′P′ is $4b\sin\frac{1}{2}\phi$; (2) the difference of the arcs AC, A′C is $4a$; (3) the locus of the intersection of the tangents at P, P′ is a cissoid; (4) taking the origin O at the single focus, and the equation $r^2-2r\,(a+b\cos\theta)+(b-a)^2=0$, the curve is its own inverse with respect to O, the radius of the circle of inversion being $b\sim a$; (5) if OPP′ be a chord through O so that P, P′ are inverse points, the locus of the point of intersection of the tangents at P, P′ is the cissoid $y^2(x+b-a)=(3a-b-x)^3$; (6) if we have a family of limaçons having a given single focus O and a given node S, $\text{OS}=b-a\equiv c$, then the locus of the centres of curvature at the points of maximum curvature is the cissoid $y^2(3c-x)=(x-c)^3$; and the envelope of the tangents at the points of inflexion is another cissoid $y^2(\frac{1}{3}c-x)=(x-\frac{1}{9}c)^3$, the origin in all these cases being at O, and the axis of x along OS.

Solution by Professor NASH, M.A.; SARAH MARKS; *and others.*

It can easily be shown that the radius of curvature is given by $\rho=\dfrac{(2bz-c^2)^{\frac{3}{2}}}{3bz-2c^2}$, where $c^2=b^2-a^2$, and thus that at the point of maximum

curvature $bz = c^2$, $\cos\theta = -\frac{a}{b}$, $\rho = c$; hence, if (r_1, θ_1), (r_2, θ_2) be the coordinates of P, P′,

$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2b}{c^2}$, therefore $2ab(\cos\theta_1\cos\theta_2 + 1) + (a^2+b^2)(\cos\theta_1 + \cos\theta_2) = 0$,

therefore $$\frac{a^2+b^2}{\cos\theta_1\cos\theta_2+1} = \frac{-2ab}{\cos\theta_1+\cos\theta_2} = \frac{(b-a)^2}{(1+\cos\theta_1)(1+\cos\theta_2)}$$
$$= \frac{(b+a)^2}{(1-\cos\theta_1)(1-\cos\theta_2)},$$

$$\therefore\ \frac{b-a}{\cos\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_2} = \frac{b+a}{\sin\frac{1}{2}\theta_1\sin\frac{1}{2}\theta_2} = \frac{2b}{\cos\frac{1}{2}(\theta_2-\theta_1)} = \frac{-2a}{\cos\frac{1}{2}(\theta_2+\theta_1)} \dots\text{(A)},$$

the positive sign being taken since $b > a$, and θ_1 and θ_2 each $< \pi$.

Again, if PP′ make an $\angle\phi$ with the axis,

$$\tan\phi = \frac{a(\sin 2\theta_2 - \sin 2\theta_1) + 2b(\sin\theta_2 - \sin\theta_1)}{a(\cos 2\theta_2 - \cos 2\theta_1) + 2b(\cos\theta_2 - \cos\theta_1)}$$
$$= -\frac{a\cos(\theta_2+\theta_1)\cos\frac{1}{2}(\theta_2-\theta_1) + b\cos\frac{1}{2}(\theta_2+\theta_1)}{a\sin(\theta_2+\theta_1)\cos\frac{1}{2}(\theta_2-\theta_1) + b\sin\frac{1}{2}(\theta_2+\theta_1)}$$
$$= -\frac{\cos\frac{1}{2}(\theta_2+\theta_1)[1-\cos(\theta_2+\theta_1)]}{\sin\frac{1}{2}(\theta_2+\theta_1)[1-2\cos^2\frac{1}{2}(\theta_2+\theta_1)]} = \tan(\theta_1+\theta_2) \text{ by (A)},$$

and $$\text{AOP} - \text{A}'\text{OP}' = \theta_1 - (\pi - \theta_2) = \theta_1 + \theta_2 - \pi = \phi.$$

Differentiating the last member of (A) and eliminating out, we get

$$\frac{d\theta_1}{\sin\theta_1} = \frac{-d\theta_2}{\sin\theta_2} = \frac{d(\theta_1+\theta_2)}{-2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)};$$

and again, from (A),

$$\frac{b}{\cos\frac{1}{2}(\theta_2-\theta_1)} = \frac{-a}{\cos\frac{1}{2}(\theta_2+\theta_1)} = \frac{(a^2+b^2+2ab\cos\theta_1)^{\frac{1}{2}}}{\sin\theta_1} = \frac{(a^2+b^2+2ab\cos\theta_2)^{\frac{1}{2}}}{\sin\theta_2}.$$

$$u = \text{arc AP} - \text{arc A}'\text{P}' = \int_0^{\theta_1}(a^2+b^2+2ab\cos\theta)^{\frac{1}{2}}\,d\theta - \int_{\theta_2}^{\pi}(a^2+b^2+2ab\cos\theta)^{\frac{1}{2}}\,d\theta;$$

therefore $$\frac{du}{d\phi} = (a^2+b^2+2ab\cos\theta_1)^{\frac{1}{2}}\frac{d\theta_1}{d\phi} + (a^2+b^2+2ab\cos\theta_2)^{\frac{1}{2}}\frac{d\theta_2}{d\phi}$$
$$= \frac{-b\sin\theta_1}{\cos\frac{1}{2}(\theta_2-\theta_1)}\ \frac{\sin\theta_1}{2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)}$$
$$+ \frac{b\sin\theta_2}{\cos\frac{1}{2}(\theta_2-\theta_1)}\ \frac{\sin\theta_2}{2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)}$$
$$= \frac{b(\sin^2\theta_2 - \sin^2\theta_1)}{\sin(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)} = 2b\sin\tfrac{1}{2}(\theta_1+\theta_2) = 2b\cos\tfrac{1}{2}\phi,$$

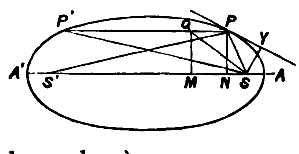
therefore $$u = 4b\sin\tfrac{1}{2}\phi,$$

Where P, P′ coincide with c, $\sin\frac{1}{2}\phi = \cos\theta$, therefore

$$\text{arc AC} - \text{arc A}'\text{C} = 4b\,\frac{a}{b} = 4a.$$

These results can be more easily obtained by inversion. If c be the radius

of inversion, the inverse of the limaçon is an ellipse whose axes are $2b$, $2c$. The point of maximum curvature is the inverse of the extremity of the minor axis, and the line PP′ is parallel to the major axis.



(1) $\mathrm{ASP} - \mathrm{A'SP'} = \mathrm{ASP} - \mathrm{AS'P} = \mathrm{SPS'}$. Inverse of line PP′ is circle SPP′S′, and the angle at which this cuts SS′ is equal to the angle in the segment SPS′.

(2) Inverse of element of arc δs is $\dfrac{c^2\delta s}{r^2}$,

therefore $u = \text{arc AP} - \text{arc A'P'}$ is $c^2\displaystyle\int\left\{\frac{1}{r^2}-\frac{1}{(2b-r)^2}\right\}ds$ over arc AP.

Now, if $\quad \mathrm{SPS'} = \phi,\ \cos\tfrac{1}{2}\phi = -\sin \mathrm{SP}y = \dfrac{c}{[r\,(2b-r)]^{\frac{1}{2}}} = \ldots\ldots\ldots\ldots$(B),

and $\quad \sin\tfrac{1}{2}\phi = \cos \mathrm{SP}y = \dfrac{dr}{ds}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(C).

Differentiating (B), and therefrom, by (C), we have

$$\tfrac{1}{2}\sin\tfrac{1}{2}\phi\,\frac{d\phi}{dr} = \frac{c\,(b-r)}{r^{\frac{3}{2}}\,(2b-r)^{\frac{3}{2}}},\qquad \frac{d\phi}{ds} = \frac{2c\,(b-r)}{r^{\frac{3}{2}}\,(2b-r)^{\frac{3}{2}}},$$

therefore $\quad u = c^2\displaystyle\int_0^{\phi}\frac{4b\,(b-r)}{r^2\,(2b-r)^2}\,ds = 2b\int_0^{\phi}\cos\tfrac{1}{2}\phi\,d\phi = 4b\sin\tfrac{1}{2}\phi.$

When P, P′ coincide, $\sin\tfrac{1}{2}\phi = \dfrac{a}{b}$, therefore $u = 4a$.

(3) The tangents at P, P′ invert into circles touching the ellipse at P, P′ and passing through the focus. These circles intersect in the point Q on PP′ where SQ bisects the angle PSP′. Drawing the ordinates PN, QM, P′N′, we have, successively,

$$\frac{\mathrm{MN}}{\mathrm{SP}} = \frac{\mathrm{MN'}}{\mathrm{S'P}} = \frac{2\mathrm{CM}}{\mathrm{S'P}-\mathrm{SP}} = \frac{\mathrm{CN}}{\mathrm{CA}};\ \text{ but }\ \frac{\mathrm{CS}}{\mathrm{S'P}-\mathrm{SP}} = \frac{\mathrm{CA}}{2\mathrm{CN}},$$

$$\frac{\mathrm{CS}}{\mathrm{CM}} = \frac{\mathrm{CA}^2}{\mathrm{CN}^2},\quad \frac{\mathrm{CS}}{\mathrm{CS}-\mathrm{CM}} = \frac{\mathrm{CA}^2}{\mathrm{CA}^2-\mathrm{CN}^2} = \frac{\mathrm{CB}^2}{\mathrm{PN}^2},\quad \frac{\mathrm{SM}}{\mathrm{CS}} = \frac{\mathrm{QM}^2}{\mathrm{CB}^2},$$

therefore the locus of Q is a parabola whose vertex is S and latus rectum $(\mathrm{CB})^2/\mathrm{CS}$, therefore the inverse is a cissoid.

(4) Inverting the equation $\mathrm{SP}+\mathrm{S'P} = \mathrm{AA'} = 2b$, we get for the limaçon $2b\mathrm{SP} - 2a\mathrm{OP} = c^2$, therefore the equation referred to the single focus is

$$4a^2r^2 - 4ar\,(a^2+b^2\cos^2\theta)+c^4 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(D)},$$

and the curve is its own inverse with respect to O, the radius of inversion being $c^2/2a$.

(5) Inverting OPP′, we get a circle through S′PP′S, so that the locus is the same cissoid as in (3). Its equation referred to S is $y^2(a-x) = x^3$, and referred to O it is $y^2\left(a-\dfrac{c^2}{2a}-x\right) = \left(x+\dfrac{c^2}{2a}\right)^3 \ \ldots\ldots\ldots\ldots\ldots\ldots$(E).

If we put $a = 2a'$, $\dfrac{b^2}{2a} = b'$, and therefore $\dfrac{c^2}{2a} = b'-a'$, so that (D) takes the form given in the question, (E) becomes $y^2(3a-b-x) = (x+b-a)^3$, and not $y^2(x+b-a) = (3a-b-x)^3$, as stated.

(6) Inverting, we have to find the locus of the inverse of the focus of a system of confocal conics with respect to the circle of curvature at the extremity of the minor axis, and this is easily shown to be the parabola $y^2 = cx$, the origin being the focus, and $2c$ the distance between the foci. Inverting again, c being the radius of inversion, the equation of the locus referred to S is $y^2(c-x) = x^3$. Transferring the origin to O, since $OS = \frac{1}{3}c$ in limaçon, the equation is

$$y^2\left(\tfrac{2}{3}c - x\right) = \left(x - \tfrac{1}{3}c\right)^3.$$

(7) Using the equation $r = a\cos\theta + b$ at the point of inflexion,

$$r = \frac{2c^2}{3b^2}, \qquad \cos\theta = -\frac{2a^2+b^2}{3ab}\,\sin\theta = \frac{c\,(4a^2-b^2)^{\frac{1}{2}}}{3ab}.$$

The tangent is $x\,(a\cos 2\theta + b\cos\theta) + y\,(a\sin 2\theta + b\sin\theta) = (a\cos\theta + b)^2$,

therefore the inflexional tangent is $cx\,(b^2+8a^2) + y\,(4a^2-b^2)^{\frac{3}{2}} + 4ac^3 = 0$,

or, if $$OS = k = \frac{c^2}{2a} = \frac{b^2-a^2}{2a},$$

$$(2k)^{\frac{3}{2}}\,x + (2k)^{\frac{1}{2}}\,a\,(9x+8k) + y\,(3a-2k)^{\frac{3}{2}} = 0,$$

a being variable.

Differentiating with respect to a, eliminating a, and then referring to O as origin, we have

$$y^2\left(x + \tfrac{2}{3}k\right) + \left(x + \tfrac{8}{9}k\right)^3 = 0, \quad \text{or} \quad y^2\left(x - \tfrac{1}{3}k\right) + \left(x - \tfrac{1}{9}k\right)^3 = 0.$$

6418. (By Professor MALET, M.A., F.R.S.)—Prove the following extension to surfaces of CHASLES' theorem for plane curves:—If to a surface of the class n any system of n parallel tangent planes be drawn, then the centre of mean position of their points of contact is fixed.

Solution by W. J. C. SHARP, M.A.

If $\dfrac{\lambda\alpha_1+\mu\alpha}{\lambda+\mu}$, $\dfrac{\lambda\beta_1+\mu\beta}{\lambda+\mu}$, $\dfrac{\lambda\gamma_1+\mu\gamma}{\lambda+\mu}$, $\dfrac{\lambda\delta_1+\mu\delta}{\lambda+\mu}$ be substituted for $\alpha, \beta, \gamma, \delta$ in $\Sigma = 0$, the tangential equation to a surface, and if

$$\Delta = \alpha\frac{d}{d\alpha_1} + \beta\frac{d}{d\beta_1} + \gamma\frac{d}{d\gamma_1} + \delta\frac{d}{d\delta_1},$$

the resulting equation in $\lambda : \mu$,

$$\lambda^n\Sigma_1 + \lambda^{n-1}\mu\Delta\Sigma_1 + \frac{1}{1\,.\,2}\lambda^{n-2}\mu^2\Delta^2\Sigma_1 + \&c. = 0,$$

determines the ratios of the perpendiculars upon the planes $(\alpha, \beta, \gamma, \delta)$ and $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ from points on the tangent planes drawn through the intersection of those planes, and therefore from the points of contact. Now,

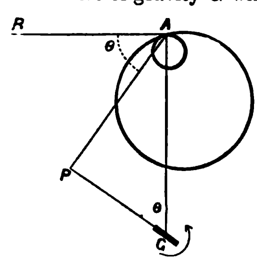
if $\Delta\Sigma_1 = 0$, or $$\alpha\frac{d\Sigma_1}{d\alpha_1} + \beta\frac{d\Sigma_1}{d\beta_1} + \gamma\frac{d\Sigma_1}{d\gamma_1} + \delta\frac{d\Sigma_1}{d\delta_1} = 0 \quad \text{.....................(1)}$$

(the equation to a point), the sum of these ratios is zero; let p, p', p'', &c. be the perpendiculars upon $(\alpha, \beta, \gamma, \delta)$, and p_1, p'_1, p''_1 those upon $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, then $\Sigma\left(\dfrac{p}{p_1}\right) = 0$, and, if $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be the plane infinity, p_1, p'_1, p''_1, &c. are equal and $\Sigma\,(p = 0)$; in this case also the tangent planes are all parallel. So that the sum of the perpendiculars from the points of contact on any plane through the point (1) is zero, and therefore that point is the mean centre of the points of contact.

5636. (By C. Leudesdorf, M.A.)—A polished uniform straight metal rod is held in a horizontal position with one end fixed at a point A, and is then allowed to swing under the action of gravity till it reaches a vertical position, when the end A is loosed, and the rod allowed to fall; find the locus traced out by the image of the fixed point A, as seen from any point by reflection at the rod during the motion of the latter.

Solution by the Rev. T. C. Simmons, M.A.

When the rod becomes vertical, it will have an angular velocity ω, which will afterwards remain constant; also its centre of gravity G will fall in a vertical line with acceleration g.



At any time t let θ be the angle the rod makes with the vertical, which will also be the angle which the perpendicular AP on its direction makes with the initial position AB, then $AG = \frac{1}{2}a + \frac{1}{2}gt^2$, $\theta = \omega t$, or, taking r, θ for the polar coordinates of the position of the image,

$$r = 2AP = 2AG\sin\theta = \left(a + \frac{g\theta^2}{\omega^2}\right)\sin\theta.$$

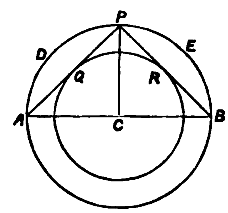
Now $\omega^2 = 3g/a$, so that gt^2 or $g\theta^2/\omega^2 = \frac{1}{3}a\theta^2$, and we have for the equation of the locus $r = (1 + \frac{1}{3}\theta^2)\, a\sin\theta$, consisting of a series of closed curves always returning through A and touching AB. They rapidly increase in size, and tend ultimately to assume the form of circles with centres lying vertically below A. The figure gives the outline of the first two branches from $\theta = 0$ to $\theta = 2\pi$, in accordance with the following table of numerical values:—

θ	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
$\dfrac{r}{a}$	·8	1·8	2·0	4·4	8·4	7·2

It may be interesting to note that the result depends only on a, the length of the rod, and is wholly independent of g. Thus the same curve would be described on the sun, or on any of the planets. The *rate* of description, however, increases with the force of gravitation.

7723. (By D. Biddle.)—If a small marble be placed at random on a circular table with raised edge, and then propelled at random in any horizontal direction; show that the probability that it will rebound from the raised edge in a direction forming an obtuse angle with the line of incidence is $\frac{1}{2} - \frac{1}{\pi} = \cdot 1816901$ or $\frac{2}{11}$ nearly.

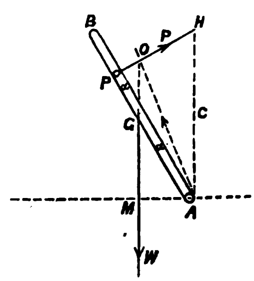
Solution by B. Reynolds, M.A.; D. Edwardes; *and others.*



The point (P) struck by the marble, may be anywhere on the circumference, and the probability as to its position is the same for the whole raised edge. But to rebound from any given position P, at an obtuse angle with the line of incidence, the marble must have been placed originally outside the chords of quadrants AP or BP. The sum of the segments cut off by these is $\frac{1}{2}\pi - 1$, hence the probability is as stated.

7693. (By Syama Charan Basu, B.A.)—A heavy rod (weight W, length $2a$) capable of free motion, in a vertical plane, about a hinge at an extremity, has a small ring sliding on it. To the ring is attached a string, which passing over a smooth pin, vertically above the hinge at a distance c, supports a weight P, hanging freely. Show that in the position of equilibrium $\tan\theta = \frac{c}{a}\left\{\left[1 + \left(\frac{Wa}{Pc}\right)^2\right]^{\frac{1}{2}} - \frac{P}{W}\right\}$, where θ is the inclination of the action on the hinge, to the horizon.

Solution by Rev. T. C. Simmons, M.A.; W. G. Lax, B.A.; *and others*



Let ϕ be the angle which the string makes with the horizon, or which the rod AB makes with the vertical; then, resolving vertically and horizontally,

$$R\sin\theta = W - P\sin\phi,$$
$$R\cos\theta = P\cos\phi,$$

whence $\tan\theta = \frac{W}{P}\sec\phi - \tan\phi.$

Again, taking moments about the hinge A,

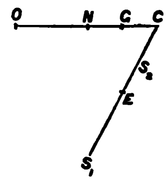
$$Wa\sin\phi = Pc\cos\phi, \quad \text{or} \quad \tan\phi = \frac{Pc}{Wa},$$

and $\tan\theta$ becomes $\frac{W}{P}\left[1 + \left(\frac{Pc}{Wa}\right)^2\right]^{\frac{1}{2}} - \frac{Pc}{Wa} = \&c.$

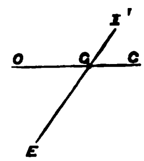
7684. (By D. Edwardes.)—Given the in-circle and circumcircle of a triangle, prove that (1) the loci of the orthocentre and centroid are circles of the respective radii, $R-2r$, $\frac{1}{3}(R-2r)$, whose centres lie on the line joining the in-centre and circumcentre, and divide it harmonically; (2) the locus of the centroid of the perimeter is a circle whose centre is collinear with the two former centres, and radius $\frac{1}{2}.(R-2r)$.

7728. (By Rev. T. C. Simmons, M.A.)—Given the circumcircle of a triangle, and any of the four circles touching the sides, show that the loci of the orthocentre and centroid are circles having the centres of the given circles as centres of similitude.

Solution by Rev. T. C. Simmons, M.A.; J. Brill, B.A.; *and others.*



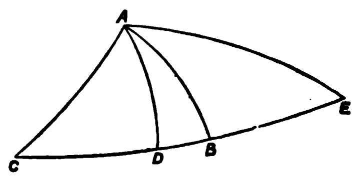
Let C be circumcentre, G centroid, N nine-point centre, O orthocentre, E the centre of the tangent circle, then, EN being constant, locus of N is a circle centre E. But $CG = \frac{2}{3}CN$, $CO = 2CN$, therefore loci of G and O are circles having, in common with the locus of N, C for one centre of similitude. Let S_1, S_2 be centres of these circles, then $[CS_2ES_1] = [CGNO]$, that is to say $[CS_1ES_2]$ is harmonic. But C is one centre of similitude, therefore E is the other.



Again, if A′, B′, C′ be the mid-points of the sides of the triangle, the centroid of its perimeter is the in-centre of A′B′C′. Also G is the common centroid, and likewise the centre of similitude of both triangles; whence, considering E originally the in-centre, and O′, I′ orthocentre and in-centre of A′B′C′, $GO' = \frac{1}{2}GO$, $GI' = \frac{1}{2}GE$, and O′ coincides with C, the original circumcentre: whence $EI' = \frac{3}{2}I'G$, so that the locus of I′ is similar and similarly situated to the locus of G, E being centre of similitude. Therefore the locus of I′ is a circle whose centre lies on EC and whose radius $= \frac{3}{2} \times \frac{1}{3}(R-2\rho) = \frac{1}{2}(R-2\rho)$.

7655. (By W. J. McClelland, B.A.)—Show that the sum of the cotangents of the intercepts made by the internal and external bisectors of the angles of a spherical triangle on the opposite sides is equal to zero.

Solution by B. H. Rau, B.A.; *the* Proposer; *and others.*



$$\cot CD \sin b$$
$$= \cot \tfrac{1}{2}A \sin C + \cos b \cos C$$
$$= (1+\cos A)\frac{\sin C}{\sin a} + \cos b \cos C;$$
$$\cot CD = \frac{\cos a \sin b + \sin C}{\sin a \sin b},$$
$$\cot CE = \frac{\cos a \sin b - \sin c}{\sin a \sin b};$$

therefore $\cot \mathrm{DE} = \dfrac{\cot \mathrm{CD} . \cot \mathrm{CE} + 1}{\cot \mathrm{CD} - \cot \mathrm{CE}}$

$$= \frac{\sin a \,.\, \sin b}{2 \sin c} \left\{ 1 + \frac{\cos^2 a \sin^2 b - \sin^2 c}{\sin^2 a \sin^2 b} \right\} = \frac{\sin^2 b - \sin^2 c}{2 \sin a \sin b \sin c}.$$

Similarly for the cotangents of the other intercepts.

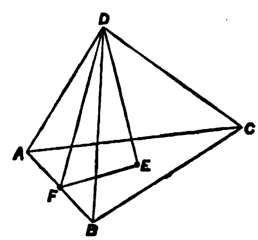
$$\text{Hence sum} = \frac{1}{2 \sin a \sin b \sin c} \left\{ \begin{array}{c} \sin^2 b - \sin^2 c + \sin^2 c - \sin^2 a \\ + \sin^2 a - \sin^2 b \end{array} \right\} = 0.$$

7656. (By W. J. C. Sharp, M.A.)—If ABCD be a tetrahedron, p_1, p_2, p_3, p_4 the perpendiculars from the vertices upon the opposite faces, then, denoting by (AB), &c., the dihedral angles between the faces which intersect in AB, &c., prove that (1)

$$\sin(\mathrm{AB}) : \sin(\mathrm{BC}) : \sin(\mathrm{CA}) : \sin(\mathrm{AD}) : \sin(\mathrm{BD}) : \sin(\mathrm{CD})$$
$$= \frac{\mathrm{AB}}{p_1 p_2} : \frac{\mathrm{BC}}{p_2 p_3} : \frac{\mathrm{CA}}{p_3 p_1} : \frac{\mathrm{AD}}{p_1 p_4} : \frac{\mathrm{BD}}{p_2 p_4} : \frac{\mathrm{CD}}{p_3 p_4};$$

and (2) the equation to the sphere described about the tetrahedron may be written $\quad ab \sin(ab)\, xy + bc \sin(bc)\, yz + \&c. = 0.$

Solution by W. G. Lax, B.A.; B. H. Rau, B.A.; *and others.*

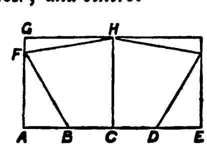


1. Draw DE, DF perpendicular to the face ABC and the edge AB; then

$$\sin(\mathrm{AB}) = \frac{\mathrm{DE}}{\mathrm{DF}} = \frac{\mathrm{AB} \,.\, \mathrm{DE}}{2 \Delta \mathrm{ABD}}$$
$$= \frac{\mathrm{AB} \,.\, p_4 \,.\, p_3}{6 \text{ vol. of tetrahedron}}$$
$$= \frac{\mathrm{AB}}{p_1 p_2} \,.\, \frac{p_1 p_2 p_3 p_4}{6\mathrm{V}} \propto \frac{\mathrm{AB}}{p_1 p_2}.$$

7636. (By Professeur Cochez.)—Inscrire dans un rectangle un pentagone ayant les côtés égaux.

Solution by B. H. Rau, B.A.; Belle Easton; *and others.*



Let AE $= a$ and AG $= b$ be the sides of the rectangle; and put BD $= x$. If the pentagon is symmetrical, the mid-point c of a side BD will coincide with the mid-point of AE, and the opposite corner of the pentagon will be at the mid-point of the opposite side of the rectangle.

AB $= \frac{1}{2}(a - x)$; therefore AF $= [x^2 - \frac{1}{4}(a-x)^2]^{\frac{1}{2}}$.

Again, GH $= \frac{1}{2}a$, therefore GF $= (x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$;

therefore $\quad b = \text{AF} + \text{FG} = [x^2 - \frac{1}{4}(a-x)^2]^{\frac{1}{2}} + (x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$,

or $\quad b^2 + x^2 - \frac{1}{4}a^2 - 2b(x^2 - \frac{1}{4}a^2) = x^2 - \frac{1}{4}(a-x)^2$,

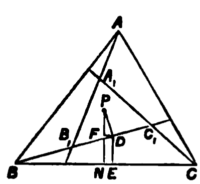
or $\quad \frac{1}{4}x^2 - \frac{1}{2}ax + b^2 = 2b(x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$,

or $\quad \frac{1}{16}x^4 - \frac{1}{4}ax^3 + (\frac{1}{4}a^2 + \frac{1}{2}b^2 - 4b^2)x^2 - ab^2x + b^4 + a^2b^2) = 0$,

a biquadratic for x.

7682. (By H. Fortey, M.A.)—Suppose three straight lines pass through the points A, B, C respectively, and turn about those points in the plane ABC with the same angular velocity and in the same direction. Find the locus of the centre of the circle described about the variable triangle thus formed, (1) when the lines through A, B, C are initially coincident with AB, BC, CA respectively; (2) when they initially coincide with AC, BA, CB; showing that the loci are two equal circles of radius $abc(\lambda - 16\Delta^2)^{\frac{1}{2}} / 16\Delta^2$ (where $\lambda = a^2b^2 + b^2c^2 + c^2a^2$ and Δ = area of ABC), that these circles touch each other at the centre of the circle about ABC, that (if $\text{A} = a^4 - b^2c^2$, &c.) the equation to the line joining their centres is $(2\text{A}+\text{B}+\text{C})\,bc\alpha + (2\text{B}+\text{C}+\text{A})\,ca\beta + (2\text{C}+\text{A}+\text{B})\,ab\gamma = 0$, and that this line touches the Brocard circle.

Solution by the Proposer.



Taking the first case; let

$$\angle \text{BAB}_1 = \text{CBC}_1 = \text{ACA}_1 = \theta,$$

and let P be the centre of the circle about $\text{A}_1\text{B}_1\text{C}_1$. Draw PN, PD perpendicular to BC, BC_1 and DE, DF perpendicular to BC, PN. Let $\text{BN} = x$, $\text{PN} = y$, also let

$$a^2 + b^2 + c^2 = k, \quad a^4 + b^4 + c^4 = \nu.$$

Now, we have

$$\text{BC}_1 = a\frac{\sin(\text{C}-\theta)}{\sin \text{C}} = a\left(\cos\theta - \frac{\cos \text{C}}{\sin \text{C}}\sin\theta\right) = a\left(\cos\theta - \frac{k-2c^2}{4\Delta}\sin\theta\right);$$

also $$\text{BB}_1 = c\,\frac{\sin\theta}{\sin \text{B}} = \frac{ac^2\sin\theta}{2\Delta};$$

$$\therefore \quad \text{BD} = a\left(\frac{\cos\theta}{2} - \frac{k-4c^2}{8\Delta}\sin\theta\right), \quad \text{B}_1\text{D} = a\left(\frac{\cos\theta}{2} - \frac{k}{8\Delta}\sin\theta\right),$$

$$\text{PD} = \text{B}_1\text{D}\cot \text{A} = \text{B}_1\text{D}\,\frac{k-2a^2}{4\Delta} = \frac{a(k-2a^2)}{4\Delta}\left(\frac{\cos\theta}{2} - \frac{k}{8\Delta}\sin\theta\right),$$

$$x = \text{BN} = \text{BE} - \text{FD} = \text{BD}\cos\theta - \text{PD}\sin\theta,$$

$$y = \text{PN} = \text{DE} + \text{PF} = \text{BD}\sin\theta + \text{PD}\cos\theta.$$

Substituting for BD and PD their values, and reducing, we get

$$\frac{\lambda - \text{A}}{16\Delta^2} - \frac{2x}{a} = \frac{b^2-c^2}{4\Delta}\sin 2\theta - \frac{a^2k-\nu}{16\Delta^2}\cos 2\theta,$$

$$\frac{2y}{a} - \frac{2c^2-a^2}{4\Delta} = \frac{b^2-c^2}{4\Delta}\cos 2\theta + \frac{a^2k-\nu}{16\Delta^2}\sin 2\theta.$$

Squaring and adding,

$$\left(\frac{2x}{a}-\frac{\lambda-A}{16\Delta^2}\right)^2+\left(\frac{2y}{a}-\frac{2c^2-a^2}{4\Delta}\right)^2=\frac{(b^2-c^2)^2}{16\Delta^2}+\frac{(a^2k+\nu)^2}{256\Delta^4},$$

or

$$\left(x-\frac{a(\lambda-A)}{32\Delta^2}\right)^2+\left(y-\frac{a(2c^2-a^2)}{8\Delta}\right)^2=\frac{a^2b^2c^2(\lambda-16\Delta^2)}{256\Delta^4},$$

the equation to a circle whose radius is

$$\frac{abc(\lambda-16\Delta^2)^{\frac{1}{2}}}{16\Delta^2}.$$

Since the ordinate of the centre of the circle is $\frac{a(2c^2-a^2)}{8\Delta}$, if α, β, γ be the trilinear coordinates of this point,

$$\frac{\alpha_1}{a(2c^2-a^2)}=\frac{\beta_1}{b(2a^2-b^2)}=\frac{\gamma_1}{c(2b^2-c^2)}.$$

If (as in the second case) the revolving lines make equal angles with AC, BA, CB respectively, then, taking C as the origin and CB as the positive direction of the axis of x, the locus would be that given above, merely modified by the interchange of b and c. It is therefore a circle of the same radius; but the ordinate of the centre is now $\frac{a(2b^2-a^2)}{8\Delta}$, and if $\alpha_2, \beta_2, \gamma_2$ are the trilinear coordinates of this centre,

$$\frac{\alpha_2}{a(2b^2-a^2)}=\frac{\beta_2}{b(2c^2-b^2)}=\frac{\gamma_2}{c(2a^2-c^2)},$$

and the equation to the line through $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$ is

$$(2A+B+C)\,bc\alpha+(2B+C+A)\,ca\beta+(2C+A+B)\,ab\gamma=0 \;\ldots\ldots(1).$$

It is easily shown that the centre of the circle about ABC lies in this line, and this centre is also clearly a point on both loci; therefore the two circles touch at that point.

Of course, the first circle passes through the first Brocard point of ABC and the second circle through the second Brocard point. Also, if ρ = radius of the Brocard circle and ω be Brocard's angle, then the radii of the above circles are each $=\rho\cot\omega$.

The equation to the line joining the Brocard points is

$$Abc\alpha+Bca\beta+Cab\gamma=0,$$

and this is parallel to (1). Therefore (1) touches the Brocard circle.

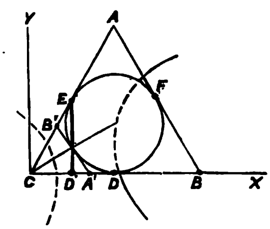
[For the Brocard circle, see *Reprint*, Vol. 40, p. 102, and *Quarterly Journal of Mathematics*, Vol. 19, p. 343.]

7385. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In an equilateral triangle ABC is inscribed a circle, any tangent to this circle meets the sides CB, CA in the points A′, B′; prove that (1) the centre of the circumscribed circle, and the centre of perpendiculars, of the triangle A′B′C have the same locus; (2) an hyperbola of which C is a focus, the centre of the circle is the farther vertex, and whose asymptotes are perpendicular to the sides CA, CB; (3) the centre of the circumscribed circle and the centre of perpendiculars are ends of a double ordinate to the transverse axis; (4) when they lie on the branch of which C is the exterior focus,

the circle is the inscribed circle of the triangle; (5) when they lie on the branch of which C is the interior focus and between the radii drawn from C parallel to the asymptotes, the circle is the escribed circle opposite C; and (6) for the remainder of that branch the circle is one of the escribed circles opposite A′ or B′.

Solution by R. KNOWLES, B.A.; SARAH MARKS; *and others.*

1. Let AC $= a$, CB′ $= y'$, CA′ $= x'$, then the coordinates of A′, B′, the centre of perpendiculars, and of the circum-centre, of A′B′C, are respectively



$$(x', 0),\quad \left(\frac{y'}{2},\ \frac{\sqrt{3}y'}{2}\right),\quad \left(\frac{y'}{2},\ \frac{2x'-y'}{2\sqrt{3}}\right),\quad \left(\frac{x'}{2},\ \frac{2y'-x'}{2\sqrt{3}}\right),$$

and $A'B'^2 = \frac{3}{4}y'^2 + (x' - \frac{1}{2}y')^2 = (a - x' - y')^2$; hence, substituting the values of the centres as above, the equation of the locus of each is $\quad x^2 + y^2 = \frac{1}{3}(a - \sqrt{3}y - 3x)^2$ ......(α).

2. This locus is an hyperbola of which C is a focus, and as the axis is perpendicular to $3x + \sqrt{3}y = a$, and passes through C, its equation is

$$y = \frac{1}{\sqrt{3}}x \quad \text{.................................}(\beta).$$

From (α), (β) we find for the vertices $x = \frac{1}{6}a$ and $\frac{1}{2}a$; hence the centre of the circle is the farther vertex, and from (α)

$$y = -\frac{1}{\sqrt{3}}\,\frac{6x^2 - 6ax + a^2}{6x - 2a} = -\frac{1}{\sqrt{3}}(x - \tfrac{2}{3}a),$$

an asymptote at right angles to AC; and $6x = 2a$ gives another at right angles to BC.

3. The equation $y + \sqrt{3}\,x = \dfrac{x' + y'}{\sqrt{3}}$ passes through the centre of perpendiculars and of the circumscribed circle, and is perpendicular to $y = \dfrac{1}{\sqrt{3}}\,x$, the transverse axis.

4. When they lie on the branch of which the centre is the vertex, A′, B′ lie on CB, CA produced, and the circle is the inscribed circle of the triangle.

5. The circle will be the escribed circle opposite C, when the line joining the two centres is within the triangle CED′, CD′ $= \frac{1}{4}a$; that is, when they lie on the branch between $x = 0$ and $y = -\dfrac{1}{\sqrt{3}}x$.

6. When A′, B′ lie on BC, AC produced, the circle is the escribed circle opposite A′ or B′.

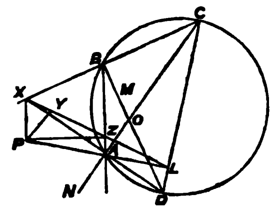
7732. (By W. J. MCCLELLAND, M.A.)—On the sides of any quadrilateral inscribed in a circle, perpendiculars are drawn from the inverse of

the point of intersection of the diagonals with respect to that circle; prove that the line of collinearity of the feet of the perpendiculars on the sides bisects at right angles the line joining the feet of the perpendiculars on the diagonals.

Solution by the Proposer; Belle Easton; *and others.*

Let ABCD be the quadrilateral, O and P inverse points, L, M, N and X, Y, Z the feet of the six perpendiculars from P on the sides and diagonals; then a circle circumscribes quadrilateral PXBM, hence

$$MX = PB \sin OBC;$$
$$NX = PC \sin OCB.$$



Therefore

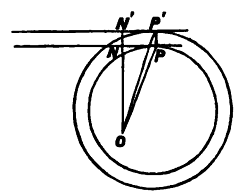
$$\frac{MX}{NX} = \frac{PB}{PC} \cdot \frac{\sin OBC}{\sin OCB} = \frac{PB}{OB} \times \frac{OC}{PC} = 1;$$

hence $MX = NX$, also $MY = NY$, $MZ = NZ$, $ML = NL$, that is, X, Y, Z, L are four points on a line, which line bisects MN at right angles.

A new Method of Deriving Legendre's Formula $\int_0^{2\pi} p\, d\omega = L$.

By Professor Cavallin, M.A.

From the origin O, within a closed convex contour, let the perpendicular ON be drawn on the tangent PN, and the perpendicular ON′ on the tangent P′N′, parallel to PN, to a curve equidistant and infinitely near to the former, so that $PP' = \mu =$ constant.



If L, S denote respectively the length and area of the given curve, S′ the area of its pedal with respect to O as origin, and p, θ the length and direction of ON, we have immediately

$$S' = S + \tfrac{1}{2}\int_0^{2\pi} (PN)^2\, d\omega = S, \quad S' = \tfrac{1}{2}\int_0^{2\pi} (ON)^2 d\omega;$$

whence, by addition, we obtain the well-known formula

$$2S' = S + \tfrac{1}{2}\int_0^{2\pi} (OP)^2\, d\omega, \quad \text{or} \quad \int_0^{2\pi} (ON)^2\, d\omega = S + \tfrac{1}{2}\int_0^{2\pi} (OP)^2\, d\omega \; \ldots (1).$$

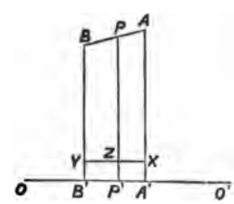
Varying (1), $$2\int_0^{2\pi} ON\, \delta ON\, d\omega = \delta S + \int_0^{2\pi} OP\, \delta OP\, d\omega \ldots\ldots\ldots\ldots (2),$$

and by substitution herein of the values $ON\, \delta ON = p\mu$, $\delta S = L\mu$, and $OP\, \delta OP = OP\,(OP' - OP) = OP\,.\,PP' \cos OP'P = \mu\, OP \sin OPN = p\mu$ in (2). Legendre's formula follows at once.

7546. (By B. Reynolds, M.A.)—Prove that, when a and b are very large compared to their difference,

$$\left(\frac{ma+nb}{m+n}\right)^{p-1} = \frac{ma^{p-1}+nb^{p-1}}{m+n}, \text{ nearly.}$$

Solution by F. C. Garner, M.A.; G. B. Mathews, B.A.; *and others.*



Draw BB′, AA′ perpendicular to any straight line OO′ and proportional to b, a respectively; and cut off B′Y, A′X so that

$$B'Y : B'B = b^{p-1} : b,$$

and $$XA' : A'A = a^{p-1} : a;$$

then, since the difference between a and b is very small, the difference between a^{p-1} and b^{p-1} is so small that it may be neglected, so that we may consider B′Y = A′X very nearly. Now take P in AB so that BP : PA = m : n; draw PP′ perpendicular to A′B′ and meeting XY in Z; then, since PP′ − BB′ is very small, we may consider

$$(PP')^{p-1} = (BB')^{p-1} \text{ very nearly;}$$

hence we may consider $ZP' = (PP')^{p-1}$ very nearly. But

$$PP' = \frac{m \,.\, AA' + n \,.\, BB'}{m+n} = \frac{ma+nb}{m+n},$$

and $$ZP' = \frac{m \,.\, A'X + n \,.\, B'Y}{m+n} = \frac{ma^{p-1}+nb^{p-1}}{m+n}; \text{ therefore, \&c.}$$

[*Otherwise*:—

$$\left(\frac{ma+nb}{m+n}\right)^{p-1} = \left(a+\frac{n(b-a)}{m+n}\right)^{p-1} = a^{p-1}\left(1+\frac{n(b-a)}{(m+n)a}\right)^{p-1}$$

$$= a^{p-1}\left(1+\frac{n}{p} \cdot \frac{b-a}{(m+n)a}\right) \text{ to first order}$$

$$= \frac{a^{p-1}}{m+n}\left\{m+n\left(1+\frac{1}{p} \cdot \frac{b-a}{a}\right)\right\}$$

$$= \frac{a^{p-1}}{m+n}\left\{m+n\left(1+\frac{b-a}{a}\right)^{p-1}\right\} \text{ to first order} = \frac{ma^{p-1}+nb^{p-1}}{m+n}.$$

As a special case, we have, when a is very nearly $= b$,

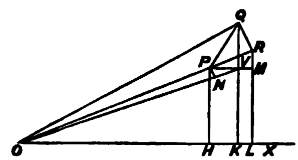
$$\left\{\tfrac{1}{3}(2a+b)\right\}^{\frac{1}{2}} = \tfrac{2}{3}a^{\frac{1}{2}} + \tfrac{1}{3}b^{\frac{1}{2}}.]$$

7739. (By W. G. Lax, B.A.)—If x, y; r, θ be the rectangular and polar coordinates of a point respectively, and if $\left(\frac{dx}{dr}\right)$ and $\left(\frac{dr}{dx}\right)$ be the

partial differential coefficients of x with respect to r, and of r with respect to x, when r, θ, and x, y are independent variables respectively; prove geometrically that $\left(\frac{dx}{dr}\right) = \left(\frac{dr}{dx}\right)$.

Solution by J. BRILL, B.A.; A. MUKHOPÁDHYÁY; *and others.*

Let $OP = r$, $OQ = r + \delta r$, $POQ = \delta\theta$; take OX as axis of x, and draw PH, QK, RL perpendicular and PM parallel to OX, and PN perpendicular to OV; then



$$\left(\frac{dx}{dr}\right) = \text{limit of } \frac{\text{HL}}{\text{PR}} = \text{limit of } \frac{\text{PM}}{\text{PR}},$$

$$\left(\frac{dr}{dx}\right) = \text{limit of } \frac{\text{NV}}{\text{HK}} = \text{limit of } \frac{\text{NV}}{\text{PV}};$$

$$\angle \text{RPM} = \text{POH}, \quad \text{also} \quad \angle \text{PVN} = \text{VOH}.$$

Now the angles POH and VOH are ultimately equal, therefore so also are the angles RPM and PVN; hence the triangles RPM and PVN are ultimately similar, therefore ultimately

$$\frac{\text{PM}}{\text{PR}} = \frac{\text{NV}}{\text{PV}}, \text{ therefore } \left(\frac{dx}{dr}\right) = \left(\frac{dr}{dx}\right).$$

7662. (By D. EDWARDES.)—Two planets being supposed to describe circular orbits of radii a, b about the Sun, with uniform velocities u, v, the planes of their orbits being at right angles; prove that, when they appear mutually stationary, their relative velocity is $\left(\frac{a^3+b^3}{a^3-b^3}\right)^{\frac{1}{2}}(v^2-u^2)^{\frac{1}{2}}$.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

If the planes of the two orbits, described by the planets A and B, be taken as those of XY, YZ, respectively, the coordinates of A taken with regard to axes through B, and moving with it, will be given by the equations $\xi = a\cos\theta$, $\eta = a\sin\theta - b\sin\phi$, $\zeta = -b\cos\phi$, where a and b are the radii of the orbits, $\theta = mt + \alpha$, and $\phi = nt + \beta$. When the planets appear mutually stationary, the area described instantaneously by either round the other is zero; therefore $\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} = 0$, and two similar equations;

hence we have $a^2m - ab(n\cos\theta\cos\phi + m\sin\theta\sin\phi) = 0$ ....(1),

$ab(m\cos\theta\cos\phi + n\sin\theta\sin\phi) - b^2n = 0$ ....(2),

$ab(m\sin\theta\cos\phi - n\cos\theta\sin\phi) = 0$ ....(3);

$n(1) + m(2)$ gives $(a^2-b^2)mn - ab(n^2-m^2)\cos\theta\cos\phi = 0$ ....(4),

but, if V be the relative velocity, we have

$$V^2 = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 = a^2m^2 + b^2n^2 - 2abmn\cos\theta\cos\phi,$$

or, by (4), $$= a^2m^2 + b^2n^2 - 2\,\frac{(a^2-b^2)\,m^2n^2}{n^2-m^2},$$

but $am = u$, $bn = v$, therefore

$$V^2 = u^2+v^2-2\,\frac{(n^2u^2-m^2v^2)}{n^2-m^2} = \frac{(m^2+n^2)(v^2-u^2)}{n^2-m^2}.$$

If the attraction follow the law of nature,

$$m^2 : n^2 = b^3 : a^3, \text{ and } V^2 = \frac{a^3+b^3}{a^3-b^3}\,(v^2-u^2).$$

It may be remarked that, in order that the planets should appear mutually stationary, the equations (1), (2), (3) must hold for the same value of t, and, if t be eliminated between them, *two* conditions will result, which must hold in order that apparently stationary positions should be possible.

6118. (By Professor Sylvester, F.R.S.)—A plane or solid reticulation, rigid but without weight, is formed by the intersections of equidistant lines or planes. One of these intersections is fixed, and at a certain number of others of them, which are given, forces may be applied. It is obvious that there are an infinite number of sets of parallel forces each containing an exact number of pound weights, which, acting at the given points of application, will balance about the fixed point.

It is required to prove that out of these a *limited* number may be selected such that by their due repetition and superposition any other balancing set whatever may be formed. In other words, i balancing sets of parallel forces P, Q ... W (i being some finite number) may be found such that any other balancing set will be made up of m_1 of the first set or its opposite, m_2 of the second set or its opposite, ... m_i of the i^{th} set or its opposite; $m_1, m_2, \ldots m_i$ being positive integers.

Solution by W. J. C. Sharp, M.A.

Beginning, as suggested by Prof. Sylvester, with the case when all the forces are in one straight line, let $P_1, P_2 \ldots P_i$; $Q_1, Q_2 \ldots Q_i$, &c., be the systems of forces, and $a_1, a_2 \ldots a_i$ the distances of the points of application from the fixed point; then, by the condition of equilibrium,

$$\left.\begin{array}{c} P_1 a_1 + P_2 a_2 + \ldots + P_i a_i = 0 \\ Q_1 a_1 + Q_2 a_2 + \ldots + Q_i a_i = 0 \\ \&c. \qquad \&c. \qquad \&c. \\ U_1 a_1 + U_2 a_2 + \ldots + U_i a_i = 0 \\ W_1 a_1 + W_2 a_2 + \ldots + W_i a_i = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, these equations being simultaneously true, it follows that quantities $m_1, m_2 \ldots m_i$ can be found such that

$$\left.\begin{array}{c} P_1 m_1 + Q_1 m_2 + \ldots + U_1 m_{i-1} + W_1 m_i = 0 \\ P_2 m_1 + Q_2 m_2 + \ldots + U_2 m_{i-1} + W_2 m_i = 0 \\ \&c. \qquad \&c. \qquad \&c. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, if all the different systems of forces be parallel, $W + m_i$ is expressed

as the resultant of multiples of the other systems, and, if the forces be so chosen that $\begin{vmatrix} P_1, & Q_1, \dots & U_1 \\ P_2, & Q_2, \dots & U_2 \\ & \&c. \quad \&c. & \\ P_{i-1}, & Q_{i-1}, \dots & U_{i-1} \end{vmatrix} = 1, \quad m_i = 1$, &c., all the other m's are integral.

Similarly, if the points lie on a plane reticulation, and the distances of the points of application from the fixed point, measured parallel to the threads, be $a_1, a_2 \dots a_i$; and $b_1, b_2 \dots b_i$ respectively, then from equations (1) which still hold, and from the corresponding system in the b's, the same system (2) follows, and similarly for a solid reticulation. So far it has been assumed that all the separate systems of forces are parallel; if this be not the case, $3(i-1)$ sets of forces may be taken, $i-1$ sets being parallel to each of the coordinate axes, and these, with the components of any other set, will form i sets of forces parallel to each of the axes, and so the last set may, by what precedes, be expressed in terms of the $3(i-1)$ other sets.

7741. (By the late Professor CLIFFORD, F.R.S.)—The motion of a point is compounded of two simple harmonic motions at right angles to one another, which are very nearly equal in period, but whose amplitudes are slowly diminishing at a uniform rate; find the general shape of the curve which the point will describe.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

Let the motion of the particle be defined by

$$x = A(1-at)\cos(kt+\alpha),$$

$$y = B(1-at)\cos[k(1+m)t+\beta] = B(1-at)\cos(kt+kmt+\beta);$$

then
$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(kmt+\beta-\alpha) = (1-at)^2\sin^2(kmt+\beta-\alpha),$$

an ellipse whose axes are constantly changing in magnitude and position, but the ratio of which is periodic, for, if ϕ be the angle which an axis makes with X, $\tan 2\phi = [2AB\cos(kmt+\beta-\alpha)]/(A^2-B^2)$ representing an oscillatory motion, whose period $= 2\pi/km$, at the expiration of which period the equation of the curve of vibration, for the instant, becomes

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(\beta-\alpha) = \left(1-\frac{2\pi a}{km}\right)^2\sin^2(\beta-\alpha),$$

an ellipse similar and similarly placed to the initial one,

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(\beta-\alpha) = \sin^2(\beta-\alpha),$$

and thus, at the expiration of each period, the linear magnitude of the ellipse is diminished in geometrical progression, while the direction of its axis major returns to its initial position, which may be taken to be that corresponding to any instant during its motion; results which can easily be verified experimentally by the kaleidophone.

5481. (By Professor BURNSIDE, M.A.)—Trace the relation between the characteristics of a curve of the mth degree having the maximum number of double points, and the curve enveloped by the line

$$(a_0, a_1, a_2, \dots a_m)(\theta, 1)^m = 0,$$

where $a_0, a_1, a_2, \dots a_m$ are linear functions of the coordinates, and θ a variable parameter.

Solution by W. J. C. SHARP, M.A.

If the given equation $(a_0, a_1, \dots a_m)(\theta, 1)^m = 0$ be identical with

$$ax + \beta y + \gamma z = 0,$$

then
$$\lambda a = (p_0, p_1, \dots p_m)(\theta, 1)^m,\ \lambda\beta = (q_0, q_1, \dots q_m)(\theta, 1)^m,$$
$$\lambda\gamma = (r_0, r_1, \dots r_m)(\theta, 1)^m,$$

where p_0, p_1, &c. ... r_m are constant, and the envelope, the tangential equation of which is the eliminant of these three equations, will be a unicursal curve of the m^{th} order if a, β, γ be looked upon as ordinary coordinates. And therefore, when they are tangential coordinates, $i + \tau = \frac{1}{2}(m-1)(m-2)$, as in the case of a unicursal curve $k + \delta = \frac{1}{2}(m-1)(m-2)$, and, using accented letters for the unicursal curve, and unaccented for the envelope,

$$i = k',\ k = i',\ \tau = \delta',\ \delta = \tau',\ m = n',\ n = m'.$$

In fact, from the considerations above, it appears that every envelope of the class here given is the polar reciprocal, with respect to $x^2 + y^2 + z^2 = 0$ of the unicursal curve

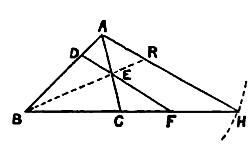
$$\lambda x = (p_0, p_1, \dots p_m)(\theta, 1)^m,\quad \lambda y = (q_0, q_1, \dots q_m)(\theta, 1)^m,$$
$$\lambda z = (r_0, r_1, \dots r_m)(\theta, 1)^m,$$

where
$$a_n \equiv p_n x + q_n y + r_n z.$$

7675. (By the EDITOR.)—Draw a transversal DEF to cut the sides AB, AC, BC of a triangle ABC in D, E, F respectively, in suchwise that, if M, N, P, Q be given lines, we shall have (1) BD : DE = M : N and CE : EF = P : Q; or (2) that BD : DE : EC = M : N : P.

Solution by W. G. LAX, B.A.; SARAH MARKS; *and others.*

1. Let (Fig. 1) a circle with centre A, and radius AH, such that CA : AH = P : Q, be drawn cutting BC produced in H; join AH; take AR so that BA : AR = M : N; join BR cutting AC in E; and draw DEF parallel to AH; then BD : DE = BA : AR = M : N, and CE : EF = CA : AH = P : Q.

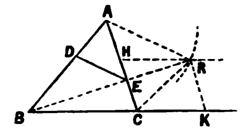


(Fig. 1.)

2. Take (Fig. 2) CH so that

$$BA : CH = M : P;$$

draw HR parallel to BC; with centre A and radius AR, such that $BA : AR = M : N$, draw a circle cutting HR in R; join AR; draw RK parallel to AC; join BR cutting AC in E; and draw DE parallel to AR; then



(Fig. 2.)

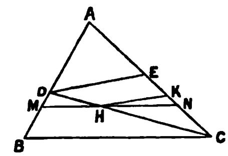
$$DE : AR = BE : BR = BD : BA \text{ and } CE : KR = BE : BR,$$

therefore $\quad BD : DE : EC = BA : AR : RK = M : N : P.$

[*Otherwise*:—Make (Fig. 3) $BM = M$ and $CK = P$; draw MN parallel to BC; inflect $KH = N$ to MN; draw CHD to AB; and through D draw DE parallel to HK; then

$$CD : CH = BD : BM = DE : HK = CE : CK,$$

therefore $BD : DE : EC$
$$= BM : HK : CK = M : N : P.]$$



(Fig. 3.)

7515. (By Professor Wolstenholme, M.A., Sc.D.)—If normals OP, OQ, OR be drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ from the point O (whose coordinates are X, Y), and the tangents at P, Q, R form a triangle P′Q′R′; prove that the ratio $k : 1$ of the triangles PQR, P′Q′R′ is given by

$$\left\{k^2 + (k-\tfrac{1}{2})\frac{a^2X^2 + b^2Y^2}{c^4}\right\}^2 = (\tfrac{1}{4}-k)\left(\frac{a^2X^2 - b^2Y^2}{c^4}\right)^2.$$

Solution by R. Lachlan, B.A.; Sarah Marks; *and others*.

Let α, β, γ be eccentric angles of P, Q, R: then we find

$$-2k = 1 + \cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) \text{ ....(1)};$$

also α, β, γ are three of the roots of

$$aX \sin\phi - bY\cos\phi = c^2 \sin\phi\cos\phi \text{ ....(2)},$$

i.e., of

$$c^4\cos^4\phi - 2ac^2X\cos^3\phi + (a^2X^2 + b^2Y^2 - c^4)\cos^2\phi + 2ac^2X\cos\phi - a^2X^2 = 0,$$

or of $\quad c^4\sin^4\phi + 2bc^2Y\sin^3\phi + \&c. = 0;$

and, if ϕ be the fourth root of (2), we find

$$\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta = \frac{a^2X^2 + b^2Y^2 - c^4}{c^4} - \frac{2aX}{c^2}\cos\phi + \cos^2\phi,$$

$$\sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta = \frac{a^2X^2 + b^2Y^2 - c^4}{c^4} + \frac{2bY}{c^2}\sin\phi + \sin^2\phi.$$

Thus (1) may be written

$$aX\cos\phi - bY\sin\phi = c^2\left(k + \frac{a^2X^2+b^2Y^2}{c^4}\right) \dots\dots\dots\dots\dots(3).$$

From (2) and (3), by multiplication, $-ab\,XY = c^4\,k\sin\phi\cos\phi$ ....(4), also, squaring (2) and (3), we get, by addition,

$$a^2X^2 + b^2Y^2 - 4ab\,XY\sin\phi\cos\phi = c^4\sin^2\phi\cos^2\phi + c^4\left(k + \frac{a^2X^2+b^2Y^2}{c^4}\right)^2.$$

Substituting from (4), we have, after a little reduction, the stated result.

[Since four normals can be drawn from (X, Y), there are four values of k; and, if these be k_1, k_2, k_3, k_4, the equation proves that

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} = 4,$$

$$k_1^2 + k_2^2 + k_3^3 + k_4^2 - 2(k_2k_3 + \dots) + 2(k_1 + k_2 + k_3 + k_4) = 0.$$

When $a^2X^2 = b^2Y^2$, the four points P, Q, R, S may be taken so that PQ, RS are parallel to each other and to one of the equal conjugate diameters; in which case the triangles PQR, PQS will be equal to each other, and so also P′Q′R′, P′Q′S′; PRS, QRS; P′R′S′, Q′R′S′.

The equation will also have equal roots when (X, Y) lies on the evolute. When k is positive, it is of course $< \frac{1}{4}$; and, since

$$k = -2\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta),$$

its real negative values must be > -2, or, more exactly, its value for real triangles > -2; which is the value at the centres of maximum and minimum curvature.

It may easily be proved, from the quartic in k, that any positive value of $k < u^{\frac{1}{2}}(1-u^{\frac{1}{2}})$; and any positive value of $-k < u^{\frac{1}{2}}(1-u^{\frac{1}{2}})$, u being always < 1 for real triangles, limits which are closer than $\frac{1}{4}$ and 2.]

7484. (By Professor MALET, F.R.S.)—If two solutions of the linear differential equation

$$\frac{d^3y}{dx^3} + Q_1\frac{d^2y}{dx^2} + Q_2\frac{dy}{dx} + Q_3y = 0 \dots\dots\dots\dots\dots(A)$$

are the solutions of the equation $\dfrac{d^2y}{dx^2} + P_1\dfrac{dy}{dx} + P_2y = 0$; prove that (1)

$$P_1P_2(P_1 - Q_1) = P_2\left(\frac{dP_1}{dx} + P_2 - Q_2\right) = P_1\left(\frac{dP_2}{dx} - Q_3\right);$$

and (2) the complete solution of (A) is the solution of

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = CP_2e^{-\int\frac{Q_3}{P_2}dx}.$$

Solution by R. LACHLAN, B.A.; NILKANTA SARKAR, B.A.; *and others.*

If U, V, W be any particular solutions of (A), the general solution of A is of the form $y = C_1U + C_2V + C_3W$; hence, differentiating and eliminat-

ing C_1, we get $\left(\frac{d}{dx}-X_1\right)y = C_2V'+C_3W'$, where $X_1 = \frac{1}{U}\frac{dU}{dx}$. Similarly, eliminating C_2, C_3, we shall get

$$\left(\frac{d}{dx}-X_3\right)\left(\frac{d}{dx}-X_2\right)\left(\frac{d}{dx}-X_1\right)y = 0.$$

Also, if U, V be particular solutions of (B), then (B) must be of the form

$$\left(\frac{d}{dx}-X_2\right)\left(\frac{d}{dx}-X_1\right)y = 0;$$

hence (A) is of the form $\left(\frac{d}{dx}-X\right)\left(\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y\right) = 0$; thus, comparing this with (A), we find $X = P_1-Q_1 = \dfrac{P_2-Q_2+\frac{dP_1}{dx}}{P_1} = \dfrac{\frac{dP_2}{dx}-Q_3}{P_2}$, and the form of this equation shows that the solution of (A) must be

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = ce^{\int X\,dx} = cP_2e^{-\int\frac{Q_3}{P_2}dx}.$$

[Mr. LACHLAN believes this solution to be preferable to that given by Mr. RAWSON on pp. 113 to 115 of Vol. xl. of *Reprints*, from the fact that the method here employed is capable of more general application.]

7639. (By CHRISTINE LADD FRANKLIN, B.A.)—If, in a certain lot of objects, the a's are identical with the non-x's which are b's together with the y's which are non-b's, and the c's are not identical with the x's which are non-d's together with the non-y's which are d's, what relation exists between a, b, c, d?

Solution by DR. MACFARLANE; *the* PROPOSER; *and others.*

The first and second conditions are

$$a = (1-x)\,b+y\,(1-b),\qquad c+v = x\,(1-d)+(1-y)\,d+w\ldots\ldots(1, 2),$$

where v and w are determinate, but such that both cannot be null.

These equations may be written in the form

$$xb-y\,(1-b)+a-b = 0,\ x\,(1-d)-yd+d+w-c-v = 0\ldots\ldots(3, 4).$$

Multiply (3) by b and (4) by $(1-d)$,

$$xb+b\,(a-1) = 0,\ x\,(1-d)+1-d\,(w-c-v) = 0\ldots\ldots\ldots(5, 6);$$

$\therefore\ b\,(1-d)(w-c-v) = b\,(1-d)(a-1)$, *i.e.*, $b\,(1-d)[a+v = 1-c+w]\ldots(7)$.

Again, multiply (3) and (4) by $(1-b)$ and d respectively, then

$$-y\,(1-b)+a\,(1-b) = 0,\ -yd+d\,(1-c+w-v) = 0\ldots\ldots\ldots(8, 9);$$

$\therefore\quad (1-b)\,d\,[1-c+w-v = a]$, *i.e.*, $(1-b)\,d\,[a+v = 1-c+w]\ldots\ldots(10)$.

As $b(1-d)$ and $(1-b)d$ are mutually exclusive, (7) and (10) can by addition be combined into the equivalent equation

$$[b(1-d)+(1-b)d](a+v=1-c+w) \quad \text{..................(11).}$$

Of the b's which are not d or the non-b's which are d, those which are a are not identical with those which are not c.

7554. (By T. Muir, M.A., F.R.S.E.)—Prove that

$$\begin{aligned}&(C-B)(C^6-B^6)(D-A)(D^2-A^2)A^2D^2 \\ &\qquad + (C-B)(C^2-B^2)(D-A)(D^6-A^6)B^2C^2 \\ &+ (C^4-B^4)(C^3-B^3)(D^4-A^4)(D-A)AD \\ &\qquad + (C^4-B^4)(C-B)(D^4-A^4)(D^3-A^3)BC \\ &= (C^2-B^2)(C^5-B^5)(D^2-A^2)(D^3-A^3)AD \\ &\qquad + (C^2-B^2)(C^3-B^3)(D^2-A^2)(D^5-A^5)BC.\end{aligned}$$

Solution by G. B. Mathews, B.A.; Prof. Matz, M.A.; *and others.*

Dividing both sides by $(C-B)(C^2-B^2)(D-A)(D^2-A^2)$, the equation becomes

$$\begin{aligned}&(C^4+C^2B^2+B^4)A^2D^2+(D^4+D^2A^2+A^4)B^2C^2 \\ &+ (C^2+B^2)(C^2+CB+B^2)(D^2+A^2)AD \\ &\qquad + (C^2+B^2)(D^2+DA+A^2)(D^2+A^2)BC \\ &= (C^4+C^3B+C^2B^2+CB^3+B^4)(D^2+DA+A^2)AD \\ &\qquad + (D^4+D^3A+D^2A^2+DA^3+A^4)(C^2+CB+B^2)BC \quad \text{...(A).}\end{aligned}$$

Observing that $C^4+C^2B^2+B^4=(C^2-CB+B^2)(C^2+CB+B^2)$, the left-hand side may be written

$$\begin{aligned}&AD(D^2+DA+A^2)(C^4+C^2B^2+B^4)+AD.CB(C^2+CB+B^2)(D^2+A^2) \\ &+BC(C^2+CB+B^2)(D^4+D^2A^2+D^4)+BC.AD(D^2+DA+A^2)(C^2+B^2)\end{aligned}$$

and this, by adding diagonally, becomes

$$\begin{aligned}&AD(D^2+DA+A^2)[C^4+C^2B^2+B^4+BC(C^2+B^2)] \\ &+BC(C^2+CB+B^2)[D^4+D^2A^2+A^4+AD(D^2+A^2)] \\ &= \text{right-hand side of (A).}\end{aligned}$$

5501. (By Professor Ball, LL.D., F.R.S.)—If in an equation x be changed into $k+x'^{-1}$, show that any semi-invariant of the transformed will be a covariant in k of the original equation.

Solution by W. J. C. Sharp, M.A.

If $f(x) \equiv (a_0, a_1 \dots a_n)(x, 1)^n = 0$ be the given equation, the transformed will be

$$x'^n f\left(k+\frac{1}{x'}\right) \equiv f(k)x'^n + f'(k)x'^{n-1} + \frac{1}{1.2}f''(k)x^{n-2} + \&c. = 0,$$

or, say, $$(A_0, A_1 \dots A_n)(x, 1)^n = 0,$$
where $$A_0 = (a_0, a_1 \dots a_n)(k, 1)^n,$$

$$nA_1 = n(a_0, a_1, a_2 \dots a_{n-1})(k, 1)^{n-1} = \left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) A_0,$$

$$n(n-1) A_2 = n(n-1)(a_0, a_1 \dots a_{n-2})(k, 1)^{n-2}$$
$$= n\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) A_1, \&c. \&c.$$

But $$nA_1 = f'(k) = \frac{d}{dk} A_0, \quad n(n-1) A_2 = f''(k) = n \frac{d}{dk} A_1, \&c. \&c.;$$
therefore, for any function of A_0, A_1, &c., the operations
$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) \text{ and } \frac{d}{dk}$$
give identical results. Now, if $F(A_0, A_1 \dots)$ be a function of the differences of the roots α', β', &c. of the transformed equation F_0, the coefficient of the highest power of k will be a function of the differences of the roots $\alpha, \beta \dots$ of the given equation.

Since $$\alpha' - \beta' = -(\alpha - \beta) \Big/ \left\{\left(\frac{\alpha}{k} - 1\right)\left(\frac{\beta}{k} - 1\right)\right\},$$
and therefore $$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) F_0 = 0,$$
and therefore F is a covariant in k of the original equation.

[See SALMON's *Higher Algebra*, 2nd Edition, p. 119.]

7260. (By ELIZABETH BLACKWOOD.)—A pack of n different cards is laid face downwards on a table. A person names a certain card. That and all the cards above it are shown to him, and removed. He names another; and the process is repeated until there are no cards left. Find the chance that, in the course of the operation, a card was named which was (at the time) at the top of the pack.

Solution by D. BIDDLE; BELLE EASTON; *and others.*

The card named may with equal probability occupy any position in that portion of the pack not yet removed.

There are three positions which instantly decide the issue, namely, the first (or top), the last, and the last but one; for the first and the last but one equally command success, since in the latter case only one card will remain, and this it is easy to name next time. The last or lowest position is the only one that insures failure. If the card named be in the lowest position but two, an equal chance of success or failure will be left for the next trial.

If it be in the last but three, there will in the next trial be two chances of success to one of failure; and, if in the last but four, there will be five (or $2\frac{1}{2}$) chances of success to three (or $1\frac{1}{2}$) of failure. In other words,

$P_1 = 1$, $P_2 = \frac{1}{2}$, $P_3 = \frac{2}{3}$, $P_4 = \frac{5}{8}$. But, counting from the top of the pack, the probabilities attaching to the several positions are as follows :—

$$1,\ P_{n-2},\ P_{n-3},\ P_{n-4} \dots P_1,\ 0;$$

and $$P_n = \frac{1}{n}[1 + P_{n-2} + P_{n-3} + P_{n-4} + \&c.],$$

but similarly $$P_{n-1} = \frac{1}{n-1}[1 + P_{n-3} + P_{n-4} + \&c.],$$

$\therefore\ P_n = \dfrac{1}{n}[(n-1)\,P_{n-1} + P_{n-2}]$. But this resolves itself into

$$P_n = \frac{1}{n}[(n-1)\,P_{n-2} + P_{n-3}];\ \therefore\ P_{n-2} - P_{n-3} = (n-1)(P_{n-2} - P_{n-1}),$$

which shows that the probabilities are alternately greater and less, but that the differences between them rapidly become infinitesimal as n increases. Above P_{10} the probabilities are alike to six places of decimals, viz., ·633388, or rather more than $\frac{19}{30}$, each probability being resolvable, according to the foregoing statements, into n terms of the series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \&c.$$

6740. (By Professor Asaph Hall, M.A.)—Given

$$z = a \sin(x + \alpha) + b \sin(y + \beta),$$

reduce z to the form $z = D \sin(x + \alpha + y + \beta + \delta)$.

Solution by Asûtosh Mukhopâdhyây.

We have $z = a \sin(x + \alpha) + b \sin(y + \beta)$; put $a = p + q$, $b = p - q$, then

$$\begin{aligned} z &= p\,[\sin(x+\alpha) + \sin(y+\beta)] + q\,[\sin(x+\alpha) - \sin(y+\beta)] \\ &= 2p \sin\tfrac{1}{2}(x+\alpha+y+\beta)\cos\tfrac{1}{2}(x+\alpha-y-\beta) \\ &\quad + 2q \cos\tfrac{1}{2}(x+\alpha+y+\beta)\sin\tfrac{1}{2}(x+\alpha-y-\beta). \end{aligned}$$

If we determine θ, D, so that

$$2p \cos\tfrac{1}{2}(x+\alpha-y-\beta) = D\cos\tfrac{1}{2}\theta,\quad 2q \sin\tfrac{1}{2}(x+\alpha-y-\beta) = D\sin\tfrac{1}{2}\theta,$$

we have $Z = D \sin\frac{1}{2}(x + \alpha + y + \beta + \theta)$. Finally, if we determine δ from

$$\delta = \tfrac{1}{2}\theta - \tfrac{1}{2}(x+\alpha+y+\beta),\ \text{ we get }\ Z = D\sin(x+\alpha+y+\beta+\delta),$$

which is the form required.

7643. (By Rev. H. G. Day, M.A.)—A and B sit down to play for a shilling per game, the odds being $k:1$ on B; they have m and n shillings respectively, and agree to play till one is ruined: find A's chance of success.

Solution by the Proposer.

Let u_r be A's chance when he is r games ahead; then his probability of scoring the next game is $\dfrac{1}{1+k}$, and of losing it $\dfrac{k}{1+k}$; but in these cases

his chances become u_{r+1} and u_{r-1} respectively; hence

$$(1+k)\,u_r = u_{r+1} + k u_{r-1}.$$

Solving this equation, $u_r = Ck^r + C_1$; but $u_n = 1$ and $u_{-m} = 0$; therefore

$$u_r = \frac{k^r - k^{-m}}{k^n - k^{-m}} = \frac{k^{r+m}-1}{k^{n+m}-1};\ \text{and A's chance}\ u_0 = \frac{k^m-1}{k^{n+m}-1}.$$

If k is unity, the solution becomes $u_r = Cr + C_1$; but $1 = Cn + C_1$, and $0 = -Cm + C_1$, therefore $u_r = \dfrac{r+m}{n+m}$ and $u_0 = \dfrac{m}{n+m}$;

that is, the players' chances are directly as the sums they risk.

7677. (By W. E. Johnson, B.A.)—If p and n be any integers, and $\omega_1, \omega_2 \dots \omega_{n-1}$ are all the n^{th} roots of unity except unity itself, show that the remainder, when p is divided by n, is

$$F(p) \equiv \frac{n-1}{2} + \omega_1^p \frac{\omega_1}{1-\omega_1} + \omega_2^p \frac{\omega_2}{1-\omega_2} + \dots + \omega_{n-1}^p \frac{\omega_{n-1}}{1-\omega_{n-1}}.$$

Solution by B. Hanumanta Rau, M.A.; E. Rutter; *and others.*

1. If $p = mn$, the expression becomes

$$\frac{n-1}{2} + \frac{\omega_1}{1-\omega_1} + \frac{\omega_2}{1-\omega_2} + \dots + \frac{\omega_{n-1}}{1-\omega_{n-1}}.$$

If now $y = \dfrac{x}{1-x}$, where $x^n = 1$, we have $x = \dfrac{y}{1+y}$, $\therefore\ y^n = (1+y)^n$. Corresponding to $x = 1$, we have $y = \infty$; the remaining values of y are given by $xy^{n-1} + \frac{1}{2}n(n-1)\,y^{n-2} + \dots = 0$; thus the sum of roots $= -\frac{1}{2}(n-1)$; and hence, when $p = mn$, the given expression vanishes.

2. $F(p) - F(p-1) = -(\omega_1^p + \omega_2^p + \dots + \omega_{n-1}^p)$. This equals 1, unless $p = mn$, in which case it equals $-(n-1)$. Hence the proposition is completely proved.

7292. (By Dr. Curtis.)—Two heavy particles P and Q are connected by a flexible and inextensible cord, which rests on a pulley of infinitesimal radius; P is restricted to the circumference of a smooth circle, whose centre is vertically under the pulley, or, *more generally*, of a smooth Cartesian oval, one of whose foci coincides with the pulley, and whose axis is vertical; it is required to prove that the curve to which Q should be restricted, in order that equilibrium should exist for all possible positions of P and Q, is a Cartesian oval.

Solution by the PROPOSER.

Let O, the centre of the pulley, be taken as origin, the axis of y being vertical; then, by the principle of virtual velocities, if y, y' be the coordinates of P and Q. $Py + Qy' = \text{const.}$, or $y + \lambda y' = A$; also, r and r' being the radii vectores of P and Q, $r + r' = \text{const.} = B$; but, as P is restricted as defined in the question,

$$r^2 - 2Lr + My = N, \text{ or } (B-r')^2 - 2L(B-r') + M(A-\lambda y') = N,$$

which establishes a linear relation between y' and r', and therefore represents a Cartesian oval as defined.

7615. (By W. NICOLLS, B.A.)—If $u_1 + u_2 + u_3 = c$ represent a surface of revolution, the origin being the centre of revolution, and u_1, u_2, u_3 containing respectively all the terms of the first, second, and third degrees in x, y, z; prove that u_1 is perpendicular to the axis of revolution and a factor of u_3.

Solution by J. P. JOHNSTON, B.A.; ELIZABETH BLACKWOOD; *and others.*

Considering the equation of the cubic surface in cylindrical coordinates (z, r, θ) as an equation for r, and taking the axis of z as the axis of revolution, it is evident that it can only contain even powers of r since all sections perpendicular to the axis are circles. Therefore the equation is of the form $z\left[z^2 + r^2 f_1(\theta)\right] + az^2 + r^2 f_2(\theta) + bz + c = 0$, where $f_1(\theta)$ and $f(\theta)$ are quadratic functions of $\cos\theta$ and $\sin\theta$. Hence it appears that if his be put in the form $u_3 + u_2 + u_1 = c$, u_1 is perpendicular to the axis of revolution and a factor of u_3.

[Mr. NICOLLS' theorem may be slightly generalised thus:—All cubics of revolution can be written in the form $LC + C' + L = 0$, where C and C′ are cones and L a plane perpendicular to the axis of revolution.]

7445. (By C. LEUDESDORF, M.A.)—A particle, describing a circular orbit about a centre of attractive force μ (distance)$^{-3}$ tending to a point on the circumference, is disturbed by a small force f tending to the same point; prove that the variations of the diameter $(2a)$ and of the inclination to a fixed straight line in the plane (ϖ) of that diameter which passes through the centre of force are given by the equations

$$-\operatorname{cosec}(\theta-\varpi)\frac{da}{dt} = a\sec(\theta-\varpi)\frac{d\varpi}{dt} = 4fa^3\left(\frac{2}{\mu}\right)^{\frac{1}{2}}.$$

Solution by D. EDWARDES; MARGARET T. MEYER; *and others.*

If the attraction be μD^{-5} and the velocity of projection that from in-

finity, the orbit is $u = (2a)^{-1}\sec(\theta-\varpi)$, where $h^2 = \dfrac{\mu}{8a^2}$ ....(1).

Expressing that the part of $\dfrac{du}{d\theta}$ which arises from the variation of the constants vanishes, we have $\operatorname{cosec}(\theta-\varpi)\dfrac{da}{d\theta} + a\sec(\theta-\varpi)\dfrac{d\varpi}{d\theta} = 0$......(2).

Also $\dfrac{du}{d\theta} = (2a)^{-1}\dfrac{\sin(\theta-\varpi)}{\cos^2(\theta-\varpi)}$, whence

$$\frac{d^2u}{d\theta^2} = \frac{1+\sin^2(\theta-\varpi)}{2a\cos^3(\theta-\varpi)}\left(1-\frac{d\varpi}{d\theta}\right) - \frac{\sin(\theta-\varpi)}{2a^2\cos^2(\theta-\varpi)}\frac{da}{d\theta},$$

and, since for the disturbed orbit, $\dfrac{d^2u}{d\theta^2} + u - \dfrac{\mu u^3}{h^2} = \dfrac{f}{h^2u}$, this gives by (2)

$$-\frac{4a^2}{h^2}f\cos^2(\theta-\varpi) = \frac{1}{2a\cos^3(\theta-\varpi)}\frac{d\varpi}{d\theta},$$

so that $\operatorname{cosec}(\theta-\varpi)\dfrac{da}{d\theta} = -a\sec(\theta-\varpi)\dfrac{d\varpi}{d\theta} = \dfrac{8a^4f}{h^2}\cos^4(\theta-\varpi)$.

But $$\frac{d\theta}{dt} = hu^2 = \frac{1}{8a^3}\left(\frac{\mu}{2}\right)^{\frac{1}{2}}\sec^2(\theta-\varpi);$$

$$\therefore\quad \operatorname{cosec}(\theta-\varpi)\frac{da}{dt} = -a\sec(\theta-\varpi)\frac{d\varpi}{dt} = 4fa^3\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\cos^2(\theta-\varpi).$$

7498. (By A. Martin, B.A.)—If a straight line be drawn from the focus of an ellipse to make a given angle α with the tangent, show that the locus of its intersection with the tangent will be a circle which touches or falls entirely without the ellipse according as $\cos\alpha$ is less or greater than the excentricity of the ellipse.

Solution by Rev. J. L. Kitchin, M.A.; J. O'Regan; *and others.*

The equations of the tangent and of a line through the positive focus at angle α with tangent are

$$y = mx + (m^2a^2+b^2)^{\frac{1}{2}},\quad y = \frac{m+\tan\alpha}{1-m\tan\alpha}(x-ae)\ \ ....(1,\ 2),$$

therefore $$m = \frac{y-kx+aek}{ky+x-ae},\ \text{putting } k \text{ for } \tan\alpha.$$

Hence, by (1), $$y - \frac{xy-kx^2+aekx}{ky+x-ae} = \left\{a^2\left(\frac{y-kx+aek}{ky+x-ae}\right)^2 + b^2\right\}^{\frac{1}{2}},$$

or $[kx(x-ae)+y(ky-ae)]^2 = a^2(y-kx+aek)^2 + (a^2-a^2e^2)(ky+x-ae)^2$,

which becomes $k^2(x^2+y^2) - 2ae\,ky + a^2e^2 = a^2(1+k^2)$ ....(3).

For intersection of this circle with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we get

$$\frac{a^2e^2y^2}{b^2} + \frac{2aey}{k} + \frac{b^2}{k^2} = 0 = \left(\frac{aey}{b} + \frac{b}{k}\right)^2.$$

Hence the locus is a circle which intersects the ellipse in two coincident points, *i.e.*, touches the ellipse; but, since $\frac{aey}{b}+\frac{b}{k}=0$ can only be true, as regards the ellipse, as long as $y =$ or $< -b$; hence the circle touches or falls without the ellipse according as $\frac{b^2}{ae}k <$ or $> b$, or as $\frac{1}{k} <$ or $> \frac{ae}{b}$, or as $\frac{\cos\alpha}{\sin\alpha} <$ or $> \frac{ae}{b}$, or as

$$\frac{\cos\alpha}{(\sin^2\alpha+\cos^2\alpha)^{\frac{1}{2}}} < \text{ or } > \frac{ae}{\left[b^2+\frac{a^2(a^2-b^2)}{a^2}\right]^{\frac{1}{2}}} \left\{=\frac{ae}{a}=e\right\},$$

i.e., according as $\cos\alpha <$ or $> \epsilon$.

7689. (By N'IMPORTE.)—If the two roots of the equation $x^2-a_1x+a_2=0$ are whole and positive numbers, prove that (1) $\frac{1}{36}a_2(1+a_1+a_2)(1+2a_1+4a_2)$ is a whole number decomposable into the sum of a_2 squares; (2) $\frac{1}{16}a_2^2(1+a_1+a_2)^2$ is a whole number decomposable into the sum of a_2 cubes; (3) $a_2^2(1+2a_1+4a_2)$ is decomposable into the algebraic sum of $4a_2$ squares.

Solution by B. HANUMANTA RAU, M.A.; R. KNOWLES, B.A.; *and others.*

Let m, n be the roots of the equation $x^2-a_1x+a_2=0$, then $a_1=m+n$ and $a_2=mn$.

1. $\frac{1}{36}a_2(1+a_1+a_2)(1+2a_1+4a_2) = \frac{1}{36}mn(1+m)(1+n)(1+2m)(1+2n)$
$= \frac{1}{6}m(m+1)(2m+1)\,.\,\frac{1}{6}n(n+1)(2n+1)$
$= [1^2+2^2+\ldots+m^2][1^2+2^2+\ldots+n^2] =$ sum of mn squares.

2. $\frac{1}{16}a_2^2(1+a_1+a_2)^2 = \frac{1}{16}m^2n^2(1+m)^2(1+n)^2 = [\frac{1}{2}m(m+1)]^2\,.\,[\frac{1}{2}n(n+1)]^2$
$= [1^3+2^3+\ldots+m^3][1^3+2^3+\ldots+n^3] =$ sum of mn cubes.

3. The expression $m^2n^2[1+2(m+n)+4mn] = m^2(2m+1)\,.\,n^2(2n+1)$;
now $m^2(2m+1) = 2\times\frac{1}{6}[2m(2m+1)(4m+1)]-4[\frac{1}{6}m(m+1)(2m+1)]$
$= 2\,.\,1^2+2^2+2\,.\,3^2+\ldots 2(2m-1)^2+(2m)^2 =$ algebraic sum of $2m$ squares.
Thus $a_2^2(1+2a_1+4a_2) =$ (sum of $2m$ squares) $\times$ (sum of $2n$ squares)
$=$ algebraic sum of $4mn$ or $4a_2$ squares.

7563. (By Rev. T. C. SIMMONS, M.A.)—Show that the ratio of the area of a triangle inscribed in an ellipse to the area of its polar triangle depends only on θ, ϕ, ψ, the differences between the eccentric angles of the points of contact, and is equal to $2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi\cos\frac{1}{2}\psi$.

Solution by MARGARET T. MEYER; Professor NASH, M.A.; *and others.*

The points of contact are $a\cos\alpha$, $b\sin\alpha$, $a\cos\beta$, $b\sin\beta$, $a\cos\gamma$, $b\sin\gamma$, where (α, β, γ) are the eccentric angles; and the equations of the tangents are $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$, &c.

$$\text{Area of triangle in the ellipse} = \tfrac{1}{2}\begin{vmatrix} a\cos\alpha, & b\sin\alpha, & 1 \\ a\cos\beta, & b\sin\beta, & 1 \\ a\cos\gamma, & b\sin\gamma, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}ab\left[\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)\right].$$

Area of triangle contained by the three tangents is

$$\tfrac{1}{2}\begin{vmatrix} \frac{1}{a}\cos\alpha, & \frac{1}{b}\sin\alpha, & 1 \\ \frac{1}{a}\cos\beta, & \frac{1}{b}\sin\beta, & 1 \\ \frac{1}{a}\cos\gamma, & \frac{1}{b}\sin\gamma, & 1 \end{vmatrix}^2 \div \begin{vmatrix} \frac{1}{a}\cos\alpha, & \frac{1}{b}\sin\alpha \\ \frac{1}{a}\cos\beta, & \frac{1}{b}\sin\beta \end{vmatrix} \times \begin{vmatrix} \&c. \\ \&c. \end{vmatrix} \times \begin{vmatrix} \&c. \\ \&c. \end{vmatrix}$$

$$= \tfrac{1}{2}ab\,\frac{\left[\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)\right]^2}{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)};$$

therefore the ratio is
$$\frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{\sin(\alpha-\beta) + 2\sin\frac{1}{2}(\beta-\alpha)\cos\left[\frac{1}{2}(\alpha+\beta)-\gamma\right]}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{2\sin\frac{1}{2}(\alpha-\beta)\left\{\cos\frac{1}{2}(\alpha-\beta) - \cos\left[\frac{1}{2}(\alpha+\beta)-\gamma\right]\right\}}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$= 2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha) = 2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi.$$

7698. (By R. LACHLAN, B.A.)—Show that (1) four circles can be drawn cutting the sides of a triangle in angles α, β, γ respectively; (2) if their radii be ρ, ρ_1, ρ_2, ρ_3, and they cut any other straight line in angles ϕ, ϕ_1, ϕ_2, ϕ_3, then

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}, \quad \cos\phi = \cos\phi_1 + \cos\phi_2 + \cos\phi_3.$$

Solution by Rev. T. C. SIMMONS, M.A.; B. HANUMANTA RAU, M.A.; *and others.*

Consider first the circle whose centre lies *within* the triangle; let d_1, d_2, d_3 denote the distances of its centre from the sides, and ρ its radius; then

$$d_1 = \rho\cos\alpha, \quad d_2 = \rho\cos\beta, \quad d_3 = \rho\cos\gamma; \text{ but } ad_1 + bd_2 + cd_3 = 2\Delta,$$

whence
$$\frac{1}{\rho} = \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{2\Delta}.$$

So, if ρ_1 be the radius of that circle whose centre lies beyond the side a,

$$\frac{1}{\rho_1} = -\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{2\Delta},$$

with similar expressions for $\dfrac{1}{\rho_2}$ and $\dfrac{1}{\rho_3}$, hence $\dfrac{1}{\rho} = \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} + \dfrac{1}{\rho_3}$; and, since the expressions always give real values for the four radii, four circles can always be drawn.

Again, let the equation to the new line referred to the given triangle as triangle of reference be $\lambda x + \mu y + \nu z = 0$,
and let the perpendiculars on it from the four centres be respectively p, p_1, p_2, p_3; then

$$p = \frac{\lambda d_1 + \mu d_2 + \nu d_3}{\text{some denominator D}}, \quad \text{or} \quad \frac{p}{\rho} = \frac{\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma}{\text{D}};$$

Similarly $\dfrac{p_1}{\rho_1} = \dfrac{-\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma}{\text{D}}$, &c.; $\therefore \dfrac{p}{\rho} = \dfrac{p_1}{\rho_1} + \dfrac{p_2}{\rho_2} + \dfrac{p_3}{\rho_3}$;

that is, $\cos\phi = \cos\phi_1 + \cos\phi_2 + \cos\phi_3$.

7695. (By J. O'Regan.)—Two persons play for a stake, each throwing two dice. They throw in turn, A commencing. A wins if he throws 6, B if he throws 7: the game ceasing as soon as either event happens. Show that A's chance is to B's as 30 to 31.

Solution by D. Biddle; W. J. Greenstreet, B.A.; *and others.*

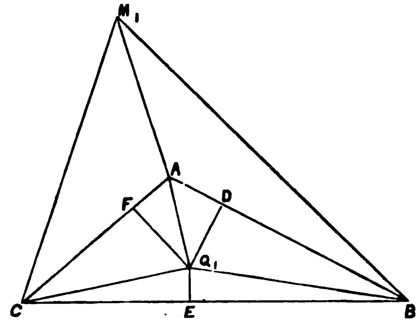
Out of 36 ways of throwing two dice, 6 may be turned up in 5 ways, viz., $1+5$, $2+4$, $3+3$, $4+2$, $5+1$; and 7 may be turned up in 6 ways, viz., $1+6$, $2+5$, $3+4$, $4+3$, $5+2$, $6+1$. There are therefore 31 chances against throwing 6, but only 30 against throwing 7. The probability that B will have a throw after A is accordingly $\frac{31}{36}$; but that A will throw again after B, only $\frac{30}{36}$.

1192. (By the Editor.)—In order to ascertain the heights of two balloons (Q, M), their angles of elevation as set forth hereunder are observed, at the same instant, from three stations (A, B, C) on the horizontal plane, whose distances apart are AB = 553, BC = 791, CA = 399 yards, (Q, A) denoting the elevation of Q at A, &c. :—

(Q, A) = 84° 10′ 10″	(M, A) = 84° 2′ 50″
(Q, B) = 76° 13′ 46·5″	(M, B) = 75° 57′ 1″
(Q, C) = 79° 35′ 5·5″	(M, C) = 79° 22′ 12″

It is also observed that only *one* of the balloons (Q) is vertically over the triangle ABC. Show that the heights of the balloons Q, M are 1874·8, 3339·4, and that their distance apart is 1560·4.

Solution by D. BIDDLE, *Member of the Aëronautical Society.*



(Scale, 400 yards to the inch.)

Let Q_1, M_1 be where plumb-lines let down from the balloons would at the moment of observation strike the earth; then

$$\left.\begin{array}{ll} \cot 84^\circ\ 10'\ 10'' & = \cdot 1021151 = \text{rel. val. of } Q_1A \\ \cot 76^\circ\ 13'\ 46\cdot 5'' & = \cdot 2450762 = \quad ,, \quad ,, \ Q_1B \\ \cot 79^\circ\ 35'\ 5\cdot 5'' & = \cdot 1838074 = \quad ,, \quad ,, \ Q_1C \end{array}\right\},$$

$$\left.\begin{array}{ll} \cot 84^\circ\ 2'\ 50'' & = \cdot 1042710 = \text{rel. val. of } M_1A \\ \cot 75^\circ\ 57'\ 1'' & = \cdot 2502499 = \quad ,, \quad ,, \ M_1B \\ \cot 79^\circ\ 22'\ 12'' & = \cdot 1876869 = \quad ,, \quad ,, \ M_1C \end{array}\right\}.$$

Moreover, since AB = 553, BC = 791, and CA = 399, therefore

$$\left(\cos A = \frac{AB^2 + CA^2 - BC^2}{2AB \, . \, CA}\right),$$

$$\begin{aligned} \cos A &= -\cdot 3640771 = \cos 111^\circ\ 21'\ 3'', \\ \cos B &= \ \ \cdot 8827700 = \cos\ 28^\circ\ 1'\ 18'', \\ \cos C &= \ \ \cdot 7589600 = \cos\ 40^\circ\ 37'\ 39''. \end{aligned}$$

Moreover, $\sin A = \cdot 9313687$; $\sin B = \cdot 4698150$; $\sin C = \cdot 6514951$.

With these data, proceeding to find the position of Q_1, draw perpendiculars therefrom to the sides of the triangle cutting AB in D, BC in E, and CA in F; then

$$AQ_1^2 - AD^2 = BQ_1^2 - (AB - AD)^2,$$

$$BQ_1^2 - BE^2 = CQ_1^2 - (BC - BE)^2, \quad CQ_1^2 - CF^2 = AQ_1^2 - (CA - CF)^2;$$

$$AD = \frac{AB}{2} - \frac{BQ_1^2 - AQ_1^2}{2AB}, \quad BD = \frac{AB}{2} + \frac{BQ_1^2 - AQ_1^2}{2AB},$$

$$BE = \frac{BC}{2} + \frac{BQ_1^2 - CQ_1^2}{2BC}, \quad CE = \frac{BC}{2} - \frac{BQ_1^2 - CQ_1^2}{2BC},$$

$$CF = \frac{AC}{2} + \frac{CQ_1^2 - AQ_1^2}{2AC}, \quad AF = \frac{AC}{2} - \frac{CQ_1^2 - AQ_1^2}{2AC}.$$

Now $$\frac{AD}{AQ_1} = \cos BAQ_1 \ \frac{AF}{AQ_1} = \cos CAQ_1,$$

$$\therefore \quad \frac{AB^2 - BQ_1^2 + AQ_1^2}{2AB \, . \, AQ_1} = \cos BAQ_1, \quad \frac{AC^2 - CQ_1^2 + AQ_1^2}{2AC \, . \, AQ_1} = \cos CAQ_1;$$

$$BQ_1 = \frac{2450762}{1021151} AQ_1 = 2{\cdot}4\, AQ_1, \quad CQ_1 = \frac{1838074}{1021151} AQ_1 = 1{\cdot}8\, AQ_1.$$

Again, $\cos BAQ_1 \cos CAQ_1 - [(1-\cos^2 BAQ_1)(1-\cos^2 CAQ_1)]^{\frac{1}{2}} = \cos A$,

$\therefore \quad 1 - \cos^2 A = \cos^2 BAQ_1 + \cos^2 CAQ_1 - 2\cos A \, . \cos BAQ_1 \, . \cos CAQ_1$.

This results in the following equation:

$$AQ_1^4 - 157898\, AQ_1^2 = -4443800000,$$

whence $AQ_1 = 348{\cdot}205$ or $191{\cdot}444$. The latter only serves, as the balloon Q is over the triangle; hence $BQ_1 = 459{\cdot}466$ and $CQ_1 = 344{\cdot}599$. Moreover, $\cdot 1021151 : 1\cdot = 191{\cdot}444 : 1874{\cdot}8$ = height of Q.

The position of M_1 and height of M can be found by the same formula in any case. But, in this,

$$BM_1 = \frac{2502499}{1042710} AM_1 = 2{\cdot}4\, AM_1, \quad CM_1 = \frac{1876869}{1042710} AM_1 = 1{\cdot}8\, AM_1,$$

which are the same proportions as in the case of Q_1. Consequently AM_1 must be the alternative or second value of AQ_1, viz., $348{\cdot}205$; also $BM_1 = 835{\cdot}692$, and $CM_1 = 626{\cdot}769$. Moreover,

$$\cdot 1042710 : 1\cdot = 348{\cdot}205 : 3339{\cdot}4 = \text{height of M}.$$

The distance between the two balloons is the hypotenuse of a right-angled triangle, of which one side is the difference of their heights and the other side is represented by M_1Q_1. To find M_1Q_1 we first find

$$\angle ACM_1 + \angle ACQ_1 = 59° \, 7' \, 25\tfrac{1}{2}''.$$

For $$\cos ACM_1 = \frac{AC^2 + CM_1^2 - AM_1^2}{2AC \, . \, CM_1} = \cos 30° \, 32' \, 11\tfrac{1}{2}'',$$

$$\cos ACQ_1 = \frac{AC^2 + CQ_1^2 - AQ_1^2}{2AC \, . \, CQ_1} = \cos 28° \, 35' \, 14''.$$

Then $CM_1 + CQ_1 : CM_1 - CQ_1 = \tan 60° \, 26' \, 17\tfrac{1}{4}'' \, . \tan \tfrac{1}{2}(CQ_1M_1 - CM_1Q_1)$,

that is, $971{\cdot}368 : 282{\cdot}170 = 1{\cdot}7627623 : \tan 27° \, 6' \, 54''$,

therefore $CQ_1M_1 = 60° \, 26' \, 17\tfrac{1}{4}'' + 27° \, 6' \, 54'' = 87° \, 33' \, 11\tfrac{1}{4}''$.

Next, $\sin CQ_1M_1 : CM_1 = \sin(ACM_1 + ACQ_1) : M_1Q_1$,

that is, $9{\cdot}9996028 : 2{\cdot}7971075 = 9{\cdot}9336278 : \log M_1Q_1$,

therefore $\log M_1Q_1 = 2{\cdot}7311325$, and $M_1Q_1 = 538{\cdot}434$. Now the difference in the heights of M and Q $= 3339{\cdot}4 - 1874{\cdot}8 = 1464{\cdot}6$; therefore $(538{\cdot}434^2 + 1464{\cdot}6^2)^{\frac{1}{2}} = MQ = 1560{\cdot}44$ = distance between the balloons.

7629. (By Belle Easton.)—A and B throw for a certain stake, A having a die whose faces are numbered 10, 13, 16, 20, 21, 25; and B a die whose faces are numbered 5, 10, 15, 20, 25, 30. If the highest throw is to win, and equal throws go for nothing; prove that the odds are 17 to 16 in favour of A.

Solution by W. J. GREENSTREET, B.A.; B. H. RAU, M.A.; *and others.*

If A throws 10, this throw is bigger than one of B's possible throws, and if A throws 13, 16, 20, 21, 25, this throw is bigger than 2, 3, 3, 4, 4 respectively; making a total of throws bigger than 17 of B's. Similarly, in B's case there is a total of throws bigger than 16 of A's. Thus the odds are 17 : 16 in A's favour.

7584. (By Rev. T. R. TERRY, M.A.)—Prove that the sum of the series whose $(p+1)^{\text{th}}$ term is $\frac{(n+p-1)!}{p!}(n-p)\,2^{n-p}$ is zero.

Solution by W. J. C. SHARP, M.A.; W. J. BARTON, B.A.; *and others.*

Denoting the series

$$(n-1)!\,nx^n+\frac{n!}{1!}(n-1)\,x^{n-1}+\frac{(n+1)!}{2!}(n-2)\,x^{n-2}+\&\text{c. by } u,$$

$$\frac{u}{x}=(n-1)!\left\{nx^{n-1}+\frac{n}{1}(n-1)\,x^{n-2}+\frac{n(n+1)}{1\,.\,2}(n-2)\,x^{n-3}+\&\text{c.}\right\},$$

$$\int\frac{u}{x}\,dx=(n-1)!\left\{x^n+\frac{n}{1}x^{n-1}+\frac{n(n+1)}{1\,.\,2}x^{n-2}+\&\text{c.}\right\}$$

$$=(n-1)!\,x^n(1-x^{-1})^{-n}=(n-1)!\left(\frac{x^2}{x-1}\right)^n;$$

$$\therefore\quad\frac{u}{x}=n!\left(\frac{x^2}{x-1}\right)^{n-1}\frac{x(x-2)}{(x-1)^2},\quad\therefore\quad u=n!\left(\frac{x^2}{x-1}\right)^n\frac{x-2}{x-1};$$

and the given series vanishes.

[The $(p+1)^{\text{th}}$ term $=\frac{(n+p-1)!}{p!}(n-p)\,.\,2^{n-p}$

$$=\frac{(n+p-1)!}{p!}\,.\,n\,.\,2^{n-p}-\frac{(n+p-1)!}{(p-1)!}\,.\,2^{n-p};$$

$$\therefore\quad\text{Sum}=n!\left\{2^n+n\,.\,2^{n-1}+\frac{n(n+1)}{1\,.\,2}\,2^{n-2}+\dots\right\}$$

$$-\,n!\left\{2^{n-1}+(n+1)\,2^{n-2}+\frac{(n+1)(n+2)}{1\,.\,2}\,.\,2^{n-3}+\dots\right\}$$

$$=n!\,.\,2^n(1-\tfrac{1}{2})^{-n}-n!\,.\,2^{n-1}(1-\tfrac{1}{2})^{-n-1}=n!\,.\,2^n\,.\,2^n-n!\,.\,2^{n-1}\,.\,2^{n+1}=0.]$$

7738. (By D. EDWARDES.) — Prove that if any three lines be drawn from the centre of a triangle ABC to meet the circum-circle in P,

Q, R, and the circle through the ex-centres in P′, Q′, R′, (1) the triangle P′Q′R′ is similar to PQR and of four times its area; (2) if the lines joining the centroid of ABC with the feet of the perpendiculars be produced through the centroid to meet the circum-circle in L, M, N, the triangle LMN is similar to the pedal triangle of ABC and of four times its area.

Solution by B. HANUMANTA RAU, M.A.; SARAH MARKS; *and others.*

1. The circum-circle of ABC is the nine-point circle of the triangle formed by joining the ex-centres.

If O, G, O′ are the circum-centre, centroid, and nine-points centre of a triangle, then OG : GO′ :: 2 : P :: R : $\frac{1}{2}$R, therefore G is the centre of similitude of the two circles. GP′, GQ′, GR′ are respectively doubles of GP, GQ, GR, therefore triangle P′Q′R′ is similar to triangle PQR and is of four times the area.

2. This follows directly from the fact that the feet of the perpendiculars of the triangle ABC are on the nine-point circle.

7737. (By N'IMPORTE.)—If all the roots of the equation $x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots \pm a_n = 0$ are whole and positive numbers, prove that (1) $a_n(1 + a_1 + \dots + a_n)(1 + 2a_1 + 4a_2 + \dots + 2^n a_n) / 6^n$ is a whole number decomposable into the sum of a_n squares; (2) $a_n^2(1 + a_1 + \dots + a_n)^2 / 4^n$ is a whole number decomposable into the sum of a_n cubes; (3) $a^2 (1 + 2a_1 + \dots + 2^n a_n$ is decomposable into the algebraic sum of $2^n a_n$ squares.

Solution by R. KNOWLES, B.A.; E. RUTTER; *and others.*

Let the n roots be $p, q, r, s \dots$, then we have

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots \pm a_n = (x-p)(x-q)(x-r)(x-s)\dots$$

Putting $x = -1$ and $-\frac{1}{2}$, we have

$$1 + a_1 + a_2 + \dots + a_n = (p+1)(q+1)(r+1)\dots$$

and $1 + 2a_1 + 4a_2 + \dots + 2^n a_n = (2p+1)(2q+1)(2r+1)\dots\ a_n = p\,.\,q\,.\,r\,.\,s.$

1. First expression $= \frac{1}{6}p(p+1)(2p+1)\,.\,\frac{1}{6}q(q+1)(2q+1)\dots$
$= (1^2 + 2^2 + \dots + p^2)(1^2 + 2^2 + \dots + q^2)(1^2 + 2^2 + \dots + r^2)$
$=$ sum of $p, q, r \dots$ squares, or a_n squares.

2. Second expression $= \frac{1}{4}p^2(p+1)^2\,.\,\frac{1}{4}q^2(q+1)^2\dots$
$= (1^3 + 2^3 + \dots + p^3)(1^3 + 2^3 + \dots + q^3)\dots =$ sum of a_n cubes.

3. Third expression $= p^2(2p+1)\,.\,q^2(2q+1)\,.\,r^2(2r+1)\dots;$

$$p^2(2p+1) = \tfrac{1}{6}[2p(2p+1)(4p+1)] - 2[\tfrac{1}{6}p(p+1)(2p+1)]$$
$$= [1^2 + 2^2 + \dots + (2p)^2] - 2[1^2 + 2^2 + \dots + p^2]$$
$$= \text{algebraical sum of } 2p \text{ squares.}$$

Hence the expression $= (2p$ squares$)(2q$ squares$) =$ sum of $2^n a_n$ squares.

1966. (By the late Samuel Bills.)—Find values of x, y that will make $S \equiv (p^2+q^2)^4 + 64\,p^2q^2(p^2-q^2)^2$ a perfect square.

Solution by Asûtosh Mukhopâdhyây.

The expression S can be at once seen to be a perfect square, when (1) $p=q$; (2) $p=0$, $S=(q^4)^2$; (3) $q=0$, $S=(p^4)^2$. Developing S, and arranging it in ascending powers of q, we have

$$S = p^8 + 68\,p^6q^2 - 122\,p^4q^4 + 68\,p^2q^6 + q^8,$$

which can easily be put into the form

$$(p^4 + 34\,p^2q^2 - 639\,q^4)^2 + 68\times 640\,p^2q^6 - [(639)^2-1]\,q^8;$$

hence, when S is a perfect square, $68\times 640\,p^2q^6 = (639^2-1)\,q^8$; whence (4) we have $p^2 = \dfrac{319}{34}q^2$.

Verification.—Substituting for p^2 this value, we get

$$S = q^8\left\{\left(\frac{353}{34}\right)^4 + \frac{34\times 64\times 319}{34^2}\left(\frac{285}{34}\right)^2\right\}$$

$$= \frac{71909249281}{34^4}q^8 = \frac{(268159)^2}{34^4}q^8;$$

therefore $$p = \sqrt{\frac{319}{34}}\,q \text{ gives } S = \left\{\frac{268159}{1156}q^4\right\}^2.$$

Similarly, if we arrange S in descending powers of q, from the symmetry of the expression we at once see the condition in this case to be $q^2 = \frac{319}{34}p^2$, and the value of S is easily inferred from (4) to be

$$\frac{(268159)^2}{34^4}p^8 = \left\{\frac{268159}{34^2}\,\frac{34^2}{319^2}q^4\right\}^2 = \left\{\frac{268159}{101761}q^4\right\}^2.$$

Note on Biot's Formula. *By* Asûtosh Mukhopâdhyây.

1. A magnetic needle suspended on its centre of gravity is constrained to move in a vertical plane, making an angle ψ with the magnetic meridian; θ is the dip in the magnetic meridian, and ϕ the dip in the vertical plane in which we suppose the motion to be.

Refer the system to rectangular axes, x being at right angles to the plane of the magnetic meridian, y horizontal and in that plane, and z vertical. Then the vertical and horizontal components are $F=M\sin\theta$, $H=M\cos\theta$, and the statical equation of virtual velocities is $F\delta z + H\delta y = 0$. Now, $y = a\cos\phi\cos\psi$, $z = a\sin\phi$, where a is the distance of any particle of the needle from its centre; therefore $\delta y = -a\sin\phi\cos\psi\,\delta\phi$, $\delta z = a\cos\phi\,\delta\phi$, therefore $F\cos\phi = H\sin\phi\cos\psi$, or $\sin\theta\cos\phi = \cos\theta\sin\phi\cos\psi$, therefore

$$\cos\phi = \cot\theta\cos\psi \quad\text{.................................(1)},$$

which determine ϕ and thence the position of equilibrium.

The equation $F\cos\phi = H\sin\phi\cos\psi$ gives $F = M\cos\theta\tan\phi\cos\psi$, which gives F, when M, &c. are known.

By logarithmic differentiation, $\frac{\Delta F}{F} = \frac{\Delta M}{M} - \frac{\sin\theta}{\cos\theta}\Delta\theta + \frac{\sec^2\phi}{\tan\phi}\Delta\phi$, a formula different from, but equivalent to, that of Biot.

Biot's Formula is easily obtained from the same source. For,

$$F = M\cos\theta\tan\phi\cos\psi \quad\text{and}\quad \sin\theta\cos\phi = \cos\theta\sin\phi\cos\psi,$$

therefore $\cos^2\phi - \cos^2\phi\cos^2\theta = \cos^2\theta\cos^2\psi(1-\cos^2\phi)$,

therefore $\cos^2\theta\cos^2\psi = \cos^2\phi(1-\cos^2\theta\sin^2\psi)$,

therefore $\frac{\cos\theta\cos\psi}{\cos\phi} = (1-\cos^2\theta\sin^2\psi)^{\frac{1}{2}}$,

therefore $F = M\sin\phi(1-\cos^2\theta\sin^2\psi)^{\frac{1}{2}}$. Taking the logarithmic differential, we have $\frac{\Delta F}{F} = \frac{\Delta M}{M} + \cot\phi\,\Delta\phi + \frac{1}{2}\cdot\frac{\sin 2\theta\sin^2\psi}{1-\cos^2\theta\sin^2\psi}\cdot\Delta\theta$, which is Biot's Formula.

2. When the needle is counterpoised by means of a brass weight so as to remain in equilibrium in a given position, we have only to substitute, in the statical equation, $F-\mu$ for F, where μ is a constant moment, therefore $F = \mu + M\cos\theta\tan\phi\cos\psi$. When the position is accurately horizontal, $\phi = 0$, therefore $F = \mu$ or $M\sin\theta = \mu$; and therefore, when the needle deviates from horizontality by a very small angle $\Delta\phi$, we have nearly

$$F = \mu + \mu\cot\theta\cos\psi\,\Delta\phi \quad\text{and}\quad \frac{F-\mu}{\mu} = \cot\theta\cos\psi\,\Delta\phi.$$

To make this formula available, it appears that the difficulty is to estimate ψ, and this may be accomplished as follows:—Let the needle be slightly moved in the vertical plane; then the vertical component called into action will be $F-\mu$, or $\mu\cot\theta\cos\psi\,\Delta\phi$; or, since $\Delta\phi$ is the arc described, the *pendulum force* will be $\mu\cot\theta\cos\psi$, and let T be the time of vibration. Afterwards, let the needle be so adjusted as to vibrate in a horizontal plane under the action only of the horizontal component $M\cos\theta$, and let T_1 be the time of vibration. Then we have, by the Theory of Pendulums, $\frac{T_1^2}{T^2} = \frac{\mu\cot\theta\cos\psi}{\mu\cot\theta} = \cos\psi$; hence $\frac{\Delta F}{F} = \frac{T_1^2}{T^2}\cot\theta\,\Delta\phi$.

The utility of this last formula is thus explained. If we could place the needle so as to vibrate exactly in the magnetic meridian, so much the better; in this case, $T_1 = T$, and we have simply $\Delta F / F = \cot\theta\,\Delta\phi$. But, since we can never be sure of this, the deviation may be estimated from the ratio $(T' : T)^2$.

7711. (By Professor Wolstenholme, M.A., Sc.D.)—Prove that (1) the locus of the points of contact of tangents drawn from a given point O to a series of confocal parabolas is a circular cubic, whose equation with O as origin is $r = 2a\sin\theta\cos\theta / \sin(\theta+\alpha)$, where $a = OS$, and 2α is the acute angle which SO makes with the common axis; S is the common focus, and the initial line is parallel to the straight line bisecting the acute angle between SO and the axis; (2) if, instead of a series of parabolas, we have a system of central conics with given foci S, S′, and centre C, the locus of the point of contact of tangents from a given point O is exactly the same,

where $a = SO \,.\, S'O / 2CO$, and α is the angle which OC makes with the straight line bisecting the angle SOS′, which is the initial line; (3) the shape of this cubic will be the same for all points O lying on the lemniscate whose equation (with C origin and CS initial line) is $r^2 \sin 2\alpha = c^2 \sin 2(\theta + \alpha)$, where $c = CS$, and α has the same meaning as before; and the foci of these lemniscates lie on the rectangular hyperbola whose foci are S, S′; (4) if any circle be described with centre O, the points of intersection of common tangents to this circle and any one of the conics whose foci are S, S′ is also this cubic, a remarkable instance of a definite locus of points, whose position (appearing to depend on two variable parameters) would be expected to be arbitrary.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

(1) Let OT be any tangent to the parabola, OC and TR parallel to the axis SV, and OV the *initial* line, then

$$\phi = \angle STO = \angle RTL = \angle COT,$$

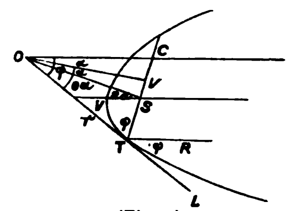
$$\therefore \quad \angle OSC = \phi + \angle SOT$$

$$= \phi + \phi - 2\alpha = 2(\phi - \alpha) = \theta,$$

and $\quad r \sin\phi = OS \,.\, \sin OSC = a \sin 2\theta,$

or $\quad r \sin(\theta + \alpha) = 2a \sin\theta \cos\theta,$

therefore $\quad r = 2a \sin\theta \cos\theta / \sin(\theta + \alpha).$

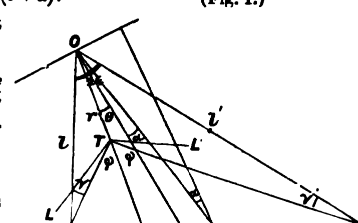


(Fig. 1.)

(2) and (5) If OT be a tangent to any conic of the confocal system, T being the point of contact, OT will be the bisector of the angle STS′, and if TL, TL′ be two tangents to any other conic of the system,

$$\angle STL = \angle S'TL',$$

and, therefore, OT also bisects $\angle LTL'$, therefore TL, TL′ will each be equidistant from O, and therefore touch a circle with O as centre, so that the loci defined in (1) and in (5) are one and the same, and will be the curve locus of point T such that the line bisecting the $\angle STS'$ shall pass through the fixed point O. This may be determined as follows:—



(Fig. 2.)

Let $\angle SOS' = 2\epsilon$, $\angle OST = \gamma$, $\angle OS'T = \gamma'$, $OS = l$, $OS' = l'$, $\angle STD = \phi = \angle S'TD$, then $l \sin\gamma = r \sin\phi = l' \sin\gamma'$, $\gamma - \gamma' = 2\theta$, $\gamma + \gamma' + 2\epsilon = 2\phi$, hence $\epsilon - \theta = \phi - \gamma$, $\epsilon + \theta = \phi - \gamma'$, therefore

$$l \sin(\epsilon - \theta) = l \sin(\phi - \gamma) = l(\sin\phi \cos\gamma - \cos\phi \sin\gamma),$$

$$l' \sin(\epsilon + \theta) = l' \sin(\phi - \gamma') = l'(\sin\phi \cos\gamma' + \cos\phi \sin\gamma'),$$

$$\therefore \quad l' \sin(\epsilon + \theta) - l \sin(\epsilon - \theta) = (l' \cos\gamma' - l \cos\gamma) \sin\phi$$

$$= \frac{ll'}{r}(\sin\gamma \cos\gamma' - \cos\gamma \sin\gamma') = \frac{ll'}{r} \sin(\gamma - \gamma') = \frac{ll'}{r} \sin 2\theta,$$

or $$\frac{2ll'}{r} \sin\theta \cos\theta = (l' + l) \cos\epsilon \sin\theta + (l' - l) \sin\epsilon \cos\theta;$$

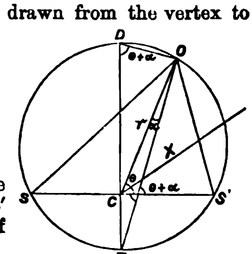
but it is geometrically evident that

$$(l'+l)\cos\epsilon = OC\cos\alpha,\quad (l'-l)\sin\epsilon = OC\sin\alpha,$$

$$\therefore \quad \frac{2ll'}{r}\sin\theta\cos\theta = 2OC(\cos\alpha\sin\theta+\sin\alpha\cos\theta) = 2OC\sin(\theta+\alpha),$$

therefore
$$r = \frac{ll'}{OC}\,\frac{\sin\theta\cos\theta}{\sin(\theta+\alpha)} = \frac{2a\sin\theta\cos\theta}{\sin(\theta+\alpha)}.$$

3. The *shape* is invariable, though the *magnitude* of the curve varies with a, so long as α is constant, and we must find the locus of the vertex of a triangle the base of which is given and the bisector of whose vertical angle makes a constant angle α with the line drawn from the vertex to the mid-point of the base: by figure,



(Fig. 3.)

$$\frac{r^2}{c^2} = \frac{CO^2}{CS.CS'} = \frac{CO^2}{CD.CE} = \frac{CO}{CD}\cdot\frac{CO}{CE}$$

$$= \frac{\sin(\theta+\alpha)}{\cos\alpha}\,\frac{\cos(\theta+\alpha)}{\sin\alpha} = \frac{\sin 2(\theta+\alpha)}{\sin 2\alpha},$$

or $r^2\sin 2\alpha = c^2\sin 2(\theta+\alpha)$, or, taking for line of reference X a line through C inclined to SS' at $\frac{1}{4}\pi$, we have $\theta = \frac{1}{4}\pi+\theta'$, and the equation of the locus becomes $r^2 = \dfrac{c^2}{\sin 2\alpha}\cos 2(\theta'+\alpha)$, the lemniscata required, the polar coordinates of a focus of which being f and α, we have $f^2 = \dfrac{c^2}{2\sin 2\alpha} = \dfrac{c^2}{4\sin\alpha\cos\alpha}$, therefore the locus of the foci, when α varies, referred to X and a line through C perpendicular to it is, $xy = \frac{1}{4}c^2$, an equilateral hyperbola whose foci are S and S'.

Again, it may be remarked that, as a tangent from any point to a conic is, at the point of contact, normal to a confocal conic, the same circular cubic will be the locus of the extremities of all *normals* drawn from a given point O to a system of confocals.

The equation of this cubic may also be obtained by combining the equation of the pair of tangents to a conic from a point, referred to *elliptic coordinates*, with the equation of the polar of the point similarly referred.

7350. (By Dr. Curtis.)—If there be circumscribed to a given conic a polygon of m sides, such that the arcs between the consecutive points of contact subtend equal angles at a focus, and 2α denote the angle which the axis of the conic makes with the radius vector drawn to any one of the points of contact, prove that (1) the product of the squares of the perpendiculars from the focus on the sides of the polygon varies inversely as $C-\cos 2m\alpha$, where C is a constant which becomes unity when the conic is a parabola; and (2) any symmetrical function of the positive powers of the

squares of the reciprocals of the perpendiculars of a degree inferior to the m^{th} remains constant, however the polygon may change consistently with the conditions of its construction. [An extension of Quest. 7317.]

Solution by the PROPOSER.

The following theorem is due to the late Professor MACCULLAGH :—
If any point O, taken on the circumference of a circle of radius b, be joined to all the angular points of a regular polygon of m sides inscribed in a concentric circle of radius a, and the angle subtended at the centre by the point O, and any angular point of the polygon, *e.g.*, the adjacent one A, be denoted by 2α, any symmetric function, inferior to the m^{th}, of the squares of the lines drawn from O to the vertices of the polygon is constant, however α may vary.

Let r_k denote the line joining the point θ to the k^{th} angular point in succession to the point A, then

$$z = r_k^2 = a^2+b^2-2ab\cos\left(2\alpha+\frac{2k\pi}{m}\right). \quad \therefore\ 2\cos\left(2\alpha+\frac{2k\pi}{m}\right) = \frac{a^2+b^2-z}{ab};$$

but, if $2\cos\phi = u$, $2\cos m\phi = u^m - mu^{m-2} + \dfrac{m.m-3}{1.2}u^{m-4} - \text{\&c.}$, therefore

$$2\cos(2m\alpha) = 2\cos(2m\alpha+2k\pi) = 2\cos m\left(2\alpha+\frac{2k\pi}{m}\right)$$

$$= \left(\frac{a^2+b^2-z}{ab}\right)^m - m\left(\frac{a^2+b^2-z}{ab}\right)^{m-2} + \frac{m.m-3}{1.2}(a^2+b^2-z)^{m-4} - \text{\&c.}$$

$$= Lz^m + Mz^{m-1} + Nz^{m-2} + \text{\&c.} + 2C,$$

or $$Lz^m + Mz^{m-1} + Nz^{m-2} + \text{\&c.} + 2(C-\cos 2m\alpha) = 0.$$

Now the only coefficient in this equation which contains α is the last, or absolute term, and, as all symmetric functions of the roots of this equation, of a degree inferior to the m^{th}, can be expressed in terms of the other coefficients, the theorem is proved.

Again, $L = (-1)^m \times \dfrac{1}{a^m b^m}$, and therefore the product of the roots of the equation for $z = 2a^m b^m (C - \cos 2m\alpha)$.

If, taking O as origin, we reciprocate these theorems, we obtain the theorems proposed, the reciprocal to the circle of radius a being an ellipse, parabola, or hyperbola, according as $b \lesseqgtr a$, and

$$2C = \left(\frac{a^2+b^2}{ab}\right)^m - m\left(\frac{a^2+b^2}{ab}\right)^{m-2} + \text{\&c.},$$

or $\quad 2a^m b^m C = (a^2+b^2)^m - m(a^2+b^2)^{m-2}a^2b^2 - \text{\&c.} = a^{2m} + b^{2m}$

(see MURPHY's *Theory of Equations*, p. 32), and $C = \dfrac{a^{2m}+b^{2m}}{2a^m b^m}$, which becomes unity when, and only when, $b = a$.

7804. (By Professor CAYLEY, F.R.S.)—1. If (a, b, c, f, g, h) are the six coordinates of a generating line of the quadric surface $x^2+y^2+z^2+w^2=0$, then $a=f,\ b=g,\ c=h$, or else $a=-f,\ b=-g,\ c=-h$, according as the line belongs to the one or the other system of generating lines.

2. If a plane meet the quadriquadric curve, $Ax^2+By^2+Cz^2+Dw^2=0$, $A'x^2+B'y^2+C'z^2+D'w^2=0$ in four points, and if (a, b, c, f, g, h) are the coordinates of the line through two of them, (a', b', c', f', g', h') of the line through the other two of them, then

$$af'+a'f=0,\quad bg'+b'g=0,\quad ch'+c'h=0.$$

Solution by W. J. C. SHARP, M.A.

Let $\dfrac{x-\alpha w}{a}=\dfrac{y-\beta w}{b}=\dfrac{z-\gamma w}{c}$ be the equation to the line whose coordinates are a, b, c, f, g, h, so that $f=c\beta-b\gamma,\ g=a\gamma-c\alpha,\ h=b\alpha-a\beta$, and $af+bg+ch=0$, and let $\alpha:\beta:\gamma:1$ be a point, the surface

$$\alpha^2+\beta^2+\gamma^2+1=0,$$

and, if the line be a generator, $x=\alpha w'+a\rho,\ y=\beta w'+b\rho,\ z=\gamma w'+c\rho$, and $w=w'$ will satisfy the equation to the quadric for all values of w' and ρ, and therefore, since $\alpha^2+\beta^2+\gamma^2+1=0$, $a\alpha+b\beta+c\gamma=0$; and $a^2+b^2+c^2=0$, $af+bg+ch=0$; also

$$(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)-(f^2+g^2+h^2)\equiv(a\alpha+b\beta+c\gamma)^2=0,$$

therefore $f^2+g^2+h^2=0$, therefore $(a\pm f)^2+(b\pm g)^2+(c\pm h)^2=0$ (all the double signs to be read alike), or $a=f,\ b=g,\ c=h$, or $a=-f$, $b=-g,\ c=-h$.

Also two lines of the same system cannot meet, for if

$$cy-bz+aw=0,\quad az-cx+bw=0 \text{ meet } c'y-b'z+a'w=0,$$

and $$a'z-c'x+b'w=0,$$

$$\begin{vmatrix} -c, & 0, & a, & b \\ 0, & c, & -b, & a \\ -c', & 0, & a', & b' \\ 0, & c', & -b', & a' \end{vmatrix}=0,\quad\text{or}\quad (ac'-a'c)^2+(bc'-b'c)^2=0,\quad\text{or}\quad \frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'},$$

and the lines are identical. If

$$\frac{x-\alpha w}{a}=\frac{y-\beta w}{b}=\frac{z-\gamma w}{c}\quad\text{and}\quad\frac{x-\alpha' w}{a'}=\frac{y-\beta' w}{b'}=\frac{z-\gamma' w}{c'}$$

be the lines a, b, c, f, g, h and a', b', c', f', g', h', then, if these are coplanar,

$$\begin{vmatrix} 1, & \alpha, & \beta, & \gamma \\ 0, & a, & b, & c \\ 1, & \alpha', & \beta', & \gamma' \\ 0, & a', & b', & c' \end{vmatrix}=0,\quad\text{or}\quad af'+a'f+bg'+b'g+ch'+c'h=0.$$

Now, if these lines be those in part (2) of the question, and $\alpha, \beta, \gamma, 1$, $\alpha', \beta', \gamma', 1$ be two of the points on the quadriquadric curve,

$$A\alpha^2+B\beta^2+C\gamma^2+D=0,\qquad A'\alpha^2+B'\beta^2+C'\gamma^2+D'=0$$
$$A\alpha'^2+B\beta'^2+C\gamma'^2+D=0,\qquad A'\alpha'^2+B'\beta'^2+C'\gamma'^2+D'=0,$$

and $$2(Aa\alpha+Bb\beta+Cc\gamma)\rho w+(Aa^2+Bb^2+Cc^2)\rho^2=0,$$
$$2(A'a\alpha+B'b\beta+C'c\gamma)\rho w+(A'a^2+B'b^2+C'c^2)\rho^2=0,$$

are simultaneously true, or

$$bcf(BC'-B'C)+cag(CA'-C'A)+abh(AB'-A'B)=0.$$

Similarly $b'c'f'(BC'-B'C)+c'a'g'(CA'-C'A)+a'b'h'(AB'-A'B)=0$,

therefore $\quad BC'-B'C : CA'-C'A : AB'-A'B$

$$=aa'(b'gch'-bg'c'h) : bb'(c'haf'-ch'a'f) : cc'(a'fbg'-af'b'g).$$

But the ratios of $BC'-B'C : CA'-C'A : AB'-A'B$,

which are constant throughout the curve, must be independent of the coordinates of two arbitrary lines each joining two points upon it, therefore

$$\frac{af'}{a'f}=\frac{bg'}{b'g}=\frac{ch'}{c'h}=\mu \text{ suppose,}$$

and if the lines be coplanar

$$(a'f+b'g+c'h)(1+\mu)=0 \text{ and } \mu=-1,$$

and therefore $\quad af'+a'f=0,\ bg'+b'g=0,\ ch'+c'h=0.$

7705. (By Professor SYLVESTER, F.R.S.)—Prove that to any given set of i quantities there corresponds a set of i other quantities, such that every symmetrical function of the differences of the first set is a function of all the successive power-sums from the second to the i^{th} inclusive of the second set. [Professor SYLVESTER calls Σa^{ω} an ω^{th} power-sum of a's.]

Solution by W. J. C. SHARP, M.A.

Let the first i quantities $\alpha, \beta \dots \lambda$ be the roots of the equation

$$z^i+ix_1z^{i-1}+\frac{i(i-1)}{1.2}x_2z^{i-2}+\dots=0.$$

Then, if U be a symmetrical function of the differences of α, β, &c., $\left(\frac{d}{dx_1}+2x_1\frac{d}{dx_2}+\dots\right)U=0$, and U is a function of any $(i-1)$ solutions of the system

$$dx_1=\frac{dx_2}{2x_1}=\frac{dx_3}{3x_2}=\text{\&c.} \quad \dots\dots\dots\dots(A).$$

Now, if there be i quantities $y_1, y_2, \dots y_i$ such that the system of equations

$$\left.\begin{array}{l} y_1dy_1+y_2dy_2+\dots+y_idy_i=0,\quad y_1^2dy_1+y_2^2dy_2+\dots+y_i^2dy_i=0 \\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ y_1^{i-1}dy_1+y_2^{i-1}dy_2+\dots+y_i^{i-1}dy_i=0 \end{array}\right\}\dots(B)$$

is a transformation of the system (A), U can be expressed as stated.

But, if $y_1=x_1+\alpha$, $y_2=x_1+\beta$, &c., $y_i=x_1+\lambda\dots$, any symmetrical function of the differences of $\alpha, \beta, \dots \lambda$ is a symmetrical function of the differences of $y_1, y_2 \dots y_i$, and therefore the system of equations

$$d\Sigma y_1=\frac{d\Sigma y_1y_2}{2\Sigma y_1}=\frac{d\Sigma y_1y_2y_3}{3\Sigma y_1y_2}=\text{\&c.} \quad \dots\dots\dots\dots(C)$$

will be equivalent to the system (A), and, since for these values of y_1, y_2, &c., $\Sigma y_1=0$, therefore $d.\Sigma y_1=0$, $d.\Sigma y_1y_2=0$, $d.\Sigma y_1y_2y_3=0$, &c...(D),

and it appears by differentiating Newton's formulæ that the system (D) is equivalent to the system (B), and U is a function of the power-sums of $y_1, y_2 \dots y_i$, from the second to the i^{th} inclusive.

4390. (By the EDITOR.)—Two gamesters, A and B, play together, A having the power to fix the stakes. Whenever A loses a game, he increases the last stake by a shilling for the next game, and diminishes it by a shilling after every gain. When they leave off playing, A has gained £13; and, had each won the same number of games, A would still, by following the above principle in regulating his stakes, have gained 10*s*. If the first stake be 30*s*., show that A won 15 and lost 5 games.

Solutions by (1) D. BIDDLE; (2) *the* PROPOSER.

1. A formula for solving questions of this sort is the following:—

$$(x-y)\left[a-\tfrac{1}{2}b\,(x-y-1)\right]+by=c,$$

where x = games won by A, y = games won by B, a = original stake, b = difference between successive stakes, c = total amount won by A. In the present instance, $a = 30$, $b = 1$, and, when x and y are equal, $c = 10$. Such being the case, the first term of the equation = 0, and $by = c$; hence $y = 10$, and $x+y = 20$.

Now, substituting $20-x$ for y, and 260 for c, the equation becomes

$$(2x-20)\left[30-\tfrac{1}{2}(2x-21)\right]+20-x = 260,$$

whence $x^2-50x = -525$, and $x = 15$; thus A has won 15 games, and B 5. The construction of the equation may readily be elucidated by taking the stakes won by A in one order, and those lost in the reverse order, turning about at each change of sign. We can arrange those won and lost, in any way: the result will be the same.

(2) −31	(3) +32		(12) +25		
(1) −30	(4) +31		(13) +24		
			(14) +23		
	(5) +30		(15) +22	(18) −21	(19) +22
	(6) +29		(16) +21	(17) −20	(20) +21
	(7) +28				
	(8) +27				
	(9) +26				
	(10) +25	(11) −24			
−61	+228	−24	+115	−41	+43

$$= 386-126 = 260.$$

For here $30+29+28+\dots+21 = 30+(30-1)+(30-2)+\dots+(30-9)$
$$= 10\left[30-\tfrac{1}{2}(10-1)\right] = (x-y)\left[a-\tfrac{1}{2}b\,(x-y-1)\right];$$
and $(32-31)+(31-30)+(25-24)+(22-21)+(21-20) = 5\times 1 = by$.

No matter in what order the 20 games have been won or lost, the stakes taken by A in the 10 surplus games will make a descending series from 30 to 21 inclusive; and he will have won besides a balance of 1 from each of the five pairs of games remaining.

2. *Otherwise*:—Let a be the first stake, c the sum added or subtracted, s_1 the gain of A on winning x and losing y games, and s_2 his gain if each player had won $\frac{1}{2}(x+y)$ games; then i standing, as usual, for $\sqrt{(-1)}$, the successive stakes and gains (or losses) may be represented as follows:

Stakes.	*Gains.*
a	$i^{\alpha}a$
$a-i^{\alpha}c$	$i^{\beta}a-i^{\alpha}i^{\beta}c$
$a-i^{\alpha}c-i^{\beta}c$	$i^{\gamma}a-i^{\alpha}i^{\gamma}c-i^{\beta}i^{\gamma}c$
$a-i^{\alpha}c-i^{\beta}c-i^{\gamma}c$	... ...
... ...	

where the quantities $\alpha, \beta, \gamma, \ldots$ are subject to the condition that x of them must be even and y odd, to correspond respectively to the x games that A wins, and the y games that he loses. If, therefore, we put m for the sum of the $(x+y)$ quantities $\alpha, \beta, \gamma, \ldots$, and n for the sum of the products of every two of them, we shall have $s_1 = am - cn$. Now m, n will be the coefficients of z^{x+y-1}, z^{x+y-2} in the equation $(z-1)^x(z+1)^y = 0$, which has x roots each $= +1$ and y roots each $= -1$; thus we have

$$m = x-y, \quad n = \tfrac{1}{2}\left[(x-y)^2 - (x+y)\right].$$

The values of x, y are therefore given by the equations

$$2a(x-y) + c(x+y) - c(x-y)^2 = 2s_1, \quad c(x+y) = 2s_2.$$

Adding, we get $\quad c(x-y) = a \pm \left[a^2 - 2c(s_1+s_2)\right]^{\frac{1}{2}}$,

therefore $\quad 2cx = 2s_2 + a \pm \left[a^2 - 2c(s_1+s_2)\right]^{\frac{1}{2}}$.

In the particular case proposed, we have $a = 30$, $c = 1$, $s_1 = 260$, $s_2 = 10$; hence $x = 15$, and $y = 5$, as stated in the question.

7807. (By the late Professor Townsend, F.R.S.)—A triangle in the plane of a conic being supposed self-reciprocal with respect to the curve; show that an infinite number of triangles could be at once inscribed to the conic and circumscribed to the triangle, or conversely.

Solution by W. J. C. Sharp, M.A.

If the given triangle be taken as triangle of reference, and

$$ax^2 + by^2 + cz^2 = 0$$

be the conic, then, $x - \lambda y = 0$, $y - \mu z = 0$, and $z - \nu x = 0$ being the sides of a triangle circumscribed to the triangle of reference, its vertices will be

$$1 : \mu\nu : \nu, \quad \lambda : 1 : \lambda\nu, \quad \lambda\mu : \mu : 1,$$

and therefore if it be also inscribed in the conic

$$a + b\mu^2\nu^2 + c\nu^2 = 0, \quad a\lambda^2 + b + c\lambda^2\nu^2 = 0, \quad a\lambda^2\mu^2 + b\mu^2 + c = 0 \ldots (1, 2, 3),$$

any one of which is easily deduced from the other two, *e.g.*, from (1) and (2)

$$[(a\lambda^2+b)(b\mu^2+c)-ac\lambda^2]\,\nu^2=0. \quad \text{Therefore } a\lambda^2\mu^2+b\mu^2+c=0,$$

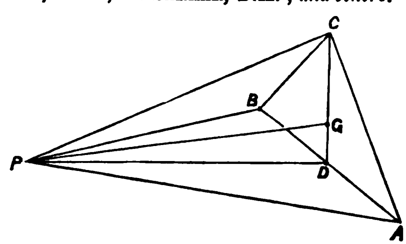
and the system of equations (1), (2), (3) admits of an infinite number of solutions.

The converse follows at once by reciprocation.

7771. (By the late Professor CLIFFORD, F.R.S.)—Find the locus of a point P which moves so that the length of the resultant of the translations PA, PB, PC is constant—the points A, B, C being fixed.

Solution by B. HANUMANTA RAU, M.A.; N. SARKAR, B.A.; *and others.*

Bisect BA at D and take G, so that CG = 2GD, then the resultant of PA, PB, PC = result. of 2PD and PC = 3PG = constant; and, since G is fixed, being the centroid of ABC, the locus of P is a sphere whose centre is the centroid of A, B, C, and radius one-third of the given resultant.



[If ρ, α, β, γ be the vectors drawn to P, A, B, C from any origin, $\alpha-\rho+\beta-\rho+\gamma-\rho$ will be the vector representing the resultant, and by the question $T(3\rho-\alpha-\beta-\gamma)=3c$ or $T[\rho-\frac{1}{3}(\alpha+\beta+\gamma)]=c$, whence the stated result follows.

7769. (By Professor SYLVESTER, F.R.S.)—Prove algebraically that, if ABC ..., A′B′C′... are two superposed projective point-series which do not possess self-conjugate points, then the segment between any two corresponding points, as AA′, BB′..., will subtend the same angle at a point properly chosen outside the line in which the point-series lie.

Solution by the PROPOSER.

Let the given line and any line perpendicular to it be assumed as axes. Call a, b, c, a', b', c', λ, λ' the distances from the origin of A, B, C, A′, B′, C′, and of the *umbra* in each series (respectively) of the infinite point in the other.

Then $(\lambda-a)(\lambda'-a')=(\lambda-b)(\lambda'-b')=(\lambda-c)(\lambda'-c')$, say $=-u$, and, if there are self-conjugate points, $(\lambda-e)(\lambda'-e)=-u$ must give real values for e. Hence, if there are not self-conjugate points, $u>[\frac{1}{2}(\lambda-\lambda')]^2$,

i.e., $\lambda\lambda'+u > [\frac{1}{2}(\lambda+\lambda')]^2$. Also, for determining λ, λ', we have the equations $(\lambda\lambda'+u)-a\lambda'-a'\lambda+aa' = 0$, $(\lambda\lambda'+u)-b\lambda'-b'\lambda+bb' = 0$,

$$(\lambda\lambda'+u)-c\lambda'-c'\lambda+cc' = 0.$$

But, if ω is the constant angle in question, we have

$$\frac{\frac{a-x}{y}-\frac{a'-x}{y}}{1-\frac{a-x}{y}\cdot\frac{a'-x}{y}} = \tan\omega, \quad \text{or, calling } y\cot\omega = v,$$

$$y^2+x^2-(a+a')x-(a-a')v+aa' = 0,$$

and similarly

$$y^2+x^2-(b+b')x-(b-b')v+bb' = 0,$$

$$y^2+x^2-(c+c')x-(c-c')v+cc' = 0.$$

Hence $y^2+x^2 = \lambda\lambda'+u$, $x+v = \lambda'$, $x-v = \lambda$, and $x = \frac{1}{2}(\lambda+\lambda')$; and x is always real, for

$$\lambda'(\lambda-a)-a'\lambda+aa' = \lambda'(\lambda-b)-b'\lambda+bb' = \lambda'(\lambda-c)-c'\lambda+cc'$$

gives

$$\frac{(a'-b')\lambda+(bb'-aa')}{b-a} = \frac{(a'-c')\lambda+(cc'-aa')}{c-a}.$$

In fact,

$$\lambda = \frac{bc(c'-b')+ca(a'-c')+ab(b'-a')}{(ab'-a'b)+(bc'-b'c)+(ca'-c'a)},$$

and

$$\lambda' = \frac{b'c'(c-b)+c'a'(a-c)+a'b'(b-a)}{(a'b-ab')+(b'c-bc')+(c'a-ca')},$$

and if $\lambda\lambda+u > [\frac{1}{2}(\lambda+\lambda')]^2$, we have $y^2+x^2 > x^2$, and y is real,

but if $\lambda\lambda+u < [\frac{1}{2}(\lambda+\lambda')]^2$, $y^2+x^2 < x^2$, and y is imaginary.

Hence the points x, y are real or imaginary according as the self-conjugate points are imaginary or real, subject to the trivial limitation that, if the two self-conjugate points are real but coincident, the point (x, y) is real, but lies on the line containing the two series, at which point the angle subtended by each segment such as AA′ will be zero.

7514. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Prove that the centroid of the arc of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, included between the positive coordinate axes, if $e = \left(1-\frac{b^2}{a^2}\right)^{\frac{1}{2}}$, and $a > b$, is given by

$$\bar{x} = \frac{a}{16e^2}\,\frac{1}{1-(1-e^2)^{\frac{3}{2}}}\left\{(3+e^2)(3e^2-1)+\frac{3(1-e^2)^3}{2e}\log\left(\frac{1+e}{1-e}\right)\right\},$$

$$\bar{y} = \frac{b}{16e^2}\,\frac{(1-e^2)^{\frac{1}{2}}}{1-(1-e^2)^{\frac{3}{2}}}\left\{(4e^2-3)(2e^2+1)+\frac{3\sin^{-1}e}{e(1-e^2)^{\frac{1}{2}}}\right\}.$$

Solution by D. EDWARDES; B. HANUMANTA RAU, M.A.; *and others.*

Putting $x = a\cos^3\phi$, $y = b\sin^3\phi$, we get

$$s = \int_0^{\frac{1}{2}\pi}\tfrac{3}{2}a\,(1-e^2\sin^2\phi)^{\frac{1}{2}}\,d\sin^2\phi = ae^{-2}\left[1-(1-e^2)^{\frac{3}{2}}\right];$$

hence $e^{-2}\left[1-(1-e^2)^{\frac{3}{2}}\right]\bar{x} = 3\int_0^{\frac{1}{2}\pi} \cos^4\phi \sin\phi\,(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}\,d\phi$

$$= 3\int_0^1 x^4\left[b^2+(a^2-b^2)\,x^2\right]^{\frac{1}{2}}dx = -\tfrac{3}{64}a^{-5}e^{-5}\int_b^{a(1-e)} \frac{(b^4-z^4)^2\,(b^2-z^2)^2}{z^7}\,dz$$

$$\left(\text{where } x = \frac{b^2-z^2}{2aez}\right)$$

$$= -\tfrac{3}{64}a^{-5}e^{-5}\left[\frac{(z^6-b^6)(z^2+b^2)(z^4-4b^2z^2+b^4)}{6z^6} + 4b^6\log z\right]_b^{a(1-e)},$$

which gives the result stated on the Question.
Also

$$ae^{-2}\left[1-(1-e^2)^{\frac{3}{2}}\right]\bar{y} = 3b\int_0^{\frac{1}{2}\pi} \sin^4\phi\cos\phi\,(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}\,d\phi$$

$$= 3ab\int_0^1 x^4\,(1-e^2x^2)^{\frac{1}{2}}\,dx = 3abe^{-5}\int_0^{\sin^{-1}e} \sin^4\theta\cos^2\theta\,d\theta$$

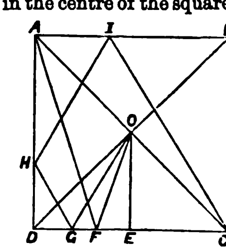
$$= 3abe^{-5}\left[\tfrac{3}{8}(\theta-\tfrac{1}{2}\sin^2\theta) - \tfrac{1}{4}\sin^3\theta\cos\theta + \tfrac{1}{6}\sin^5\theta\cos\theta - \tfrac{5}{6}\int\sin^4\theta\,d\theta\right]_0^{\sin^{-1}e}$$

$$= 3abe^{-5}\left[\tfrac{1}{16}\sin^{-1}e + \frac{e\,(1-e^2)^{\frac{1}{2}}\,(4e^2-3)(2e^2+1)}{48}\right], \text{ whence the result.}$$

5200. (By S. Tebay, B.A.)—A small marble is thrown at random on a square table having an elevated rim. If it be struck at random in any direction, determine the probability that it impinges (1) on two opposite sides; (2) on two adjacent sides, and one opposite; (3) on three consecutive sides; (4) on the four sides in succession.

Solution by D. Biddle.

The probability in each of the four cases is the ratio borne by the average sum of eight variable angles to 360°. The eight angles (two for each side of the square, since the marble can be sent from any given position against any one side in either of two directions, that is, on either side of the perpendicular) represent the limits within which the particular event specified can occur, according to the law of the equality of the angles of incidence and reflection. Let us see what the eight angles are, in each case, when the marble is originally placed in the centre of the square.



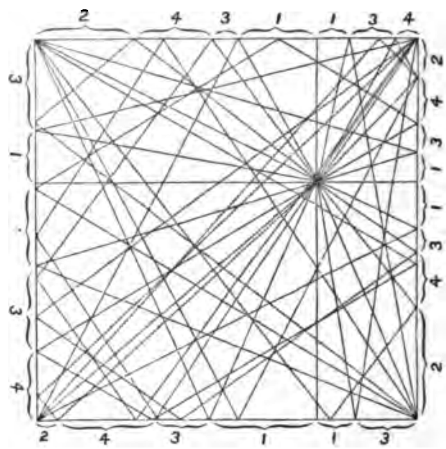
(Fig. 1.)

Here the eight angles for each case are equal and symmetrically placed. Each of them for case (1) = EOF; for case (2) = *zero*, since this case requires that the marble shall strike the first side at a less angle than 45°, which is at present impossible; for case (3) = FOG; and for case (4) = DOG. Now (taking AD = 1) EF : DF = OE : AD = $\frac{1}{2}$: 1 $\therefore$ EF = $\frac{1}{3}$ED, and DF = $\frac{1}{3}$ = tan $\angle$ EOF = tan 18° 26′. Similarly, BI = 2EG = $\frac{3}{5}$ = tan EOG = tan (FOG + EOF) = tan 30° 58′, $\therefore$ $\angle$ FOG = 12° 32′. And $\angle$ DOG = 45° − (FOG + EOF) = 14° 2′. Consequently,

when the marble happens to be in the centre of the table at the time it receives the random impulse, the probabilities are, for (1) ·40963; for (2) ·00000; for (3) ·27852; for (4) ·31185.

Let us next take an instance in which the perpendiculars divide one pair of sides into 5 and 3, and the other pair into 6 and 2. We shall then be able to construct formulæ for each of the four cases for any possible position of the marble. The brackets indicate the limiting lines in each case.

To find the tangent of the angle formed by the particular limiting line with the perpendicular, in any of the eight directions appertaining to each of the four cases, we proceed as follows:—Let a = distance of marble from particular side, b = distance of perpendicular from particular corner of table between which and perpendicular the limiting line lies. Then it is easy to see that in case (1), $x = \dfrac{b}{1+a}$; in case (2), $x = \dfrac{b}{a}$, since the limiting line extends in this



(Fig. 2.)

instance to the corner, and the compound angle, whose tangent is now under consideration, includes the individual angles of (1), (3), and (4), as well as (2); in case (3), $x = \dfrac{1+b}{2+a}$, or, when $\dfrac{1+b}{2+a} > \dfrac{b}{a}$, then $x = \dfrac{b}{a}$; and in case (4), $x = \dfrac{1+b}{1+a}$ when $a < b$, but when $a > b$, then $x = \dfrac{b}{a}$. Thus, it will be seen that (1) takes precedence of (3), (3) of (4), and (4) of (2), no tangent greater than $\dfrac{b}{a}$ being possible; and if we make equations to suit every possible instance, for the tangents of these compound angles, then for (1), $x = \dfrac{b}{a}\left(1 - \dfrac{1}{1+a}\right)$; for (3), $x = \dfrac{b}{a}\left(1 - \dfrac{2b-a}{2b+ab}\right)$; for (4), $x = \dfrac{b}{a}\left(1 - \dfrac{b-a}{b\,(1+a)}\right)$; and for (2), $x = \dfrac{b}{a}$.

In (3) and (4), if the second term has no real value [that is, if in (4) $a > b$, or in (3) $a > 2b$], then this term is discarded and $x = \dfrac{b}{a}$. Now the tangent of the compound angle formed by the limiting line with the perpendicular is, in (2), the tangent for (1) + (3) + (4) + (2); is, in (4), the tangent for (1) + (3) + (4); and is, in (3), the tangent for (1) + (3). The tangent in case (1) is the only proper tangent of the angle specially referred to by number. But it is easy to see that, if we have these several

tangents, we are virtually in possession of the angles belonging to them, and can readily arrive at the component parts by successive subtractions. The same must be true of the mean tangents, and from these we can deduce the mean angles; and, if we take the mean sum of the eight angles appertaining to each of the four cases, we have the several numerators of the probabilities required. Now a and b in the equations last given can be transposed, and, not only so, $(1-a)$ can take the place of a and $(1-b)$ of b, or $1-a$ of b and $(1-b)$ of a. In the following table are given the eight tangents for the four cases:—

	I.	II.
Perpendicular ...	a	a
Portion of Side...	b	$1-b$
(1)	$\frac{b}{a}\left(1-\frac{1}{1+a}\right)$	$\frac{1-b}{a}\left(1-\frac{1}{1+a}\right)$
(2)	$\frac{b}{a}$	$\frac{1-b}{a}$
(3)	$\frac{b}{a}\left(1-\frac{2b-a}{2b+ab}\right)$	$\frac{1-b}{a}\left(1-\frac{2-2b-a}{(1-b)(2+a)}\right)$
(4)	$\frac{b}{a}\left(1-\frac{b-a}{b+ab}\right)$	$\frac{1-b}{a}\left(1-\frac{1-b-a}{(1-b)(1+a)}\right)$

	III.	IV.
Perpendicular ...	b	b
Portion of Side...	a	$1-a$
(1)	$\frac{a}{b}\left(1-\frac{1}{1+b}\right)$	$\frac{1-a}{b}\left(1-\frac{1}{1+b}\right)$
(2)	$\frac{a}{b}$	$\frac{1-a}{b}$
(3)	$\frac{a}{b}\left(1-\frac{2a-b}{2a+ab}\right)$	$\frac{1-a}{b}\left(1-\frac{2-2a-b}{(1-a)(2+b)}\right)$
(4)	$\frac{a}{b}\left(1-\frac{a-b}{a+ab}\right)$	$\frac{1-a}{b}\left(1-\frac{1-a-b}{(1-a)(1+b)}\right)$

	V.	VI.
Perpendicular ...	$1-a$	$1-a$
Portion of Side...	b	$1-b$
(1)	$\frac{b}{1-a}\left(1-\frac{1}{2-a}\right)$	$\frac{1-b}{1-a}\left(1-\frac{1}{2-a}\right)$
(2)	$\frac{b}{1-a}$	$\frac{1-b}{1-a}$
(3)	$\frac{b}{1-a}\left(1-\frac{2b-1+a}{b\,(3-a)}\right)$	$\frac{1-b}{1-a}\left(1-\frac{a+1-2b}{(1-b)(3-a)}\right)$
(4)	$\frac{b}{1-a}\left(1-\frac{b-1+a}{b\,(2-a)}\right)$	$\frac{1-b}{1-a}\left(1-\frac{a-b}{(1-b)(2-a)}\right)$

	VII.	VIII.
Perpendicular ...	$1-b$	$1-b$
Portion of Side...	$1-a$	a
(1)	$\frac{1-a}{1-b}\left(1-\frac{1}{2-b}\right)$	$\frac{a}{1-b}\left(1-\frac{1}{2-b}\right)$
(2)	$\frac{1-a}{1-b}$	$\frac{a}{1-b}$
(3)	$\frac{1-a}{1-b}\left(1-\frac{b+1-2a}{(1-a)(3-b)}\right)$	$\frac{a}{1-b}\left(1-\frac{2a-1+b}{a\,(3-b)}\right)$
(4)	$\frac{1-a}{1-b}\left(1-\frac{b-a}{(1-a)(2-b)}\right)$	$\frac{a}{1-b}\left(1-\frac{a-1+b}{a\,(2-b)}\right)$

This table, with the aid of a table of natural tangents, will enable us to find the respective probabilities in the four cases, for any given position of the marble. To find the average probability for all positions, we must find the mean tangents. In the annexed diagram is shown in what proportion of positions cases (2) and (4) are possible. If DC be the side of the table against which the marble is sent, and D the corner between which and the perpendicular it impinges; then, to fulfil the requirements of case (2), it must have been placed originally below the diagonal DB, that is, somewhere in the triangle DBC; and, to

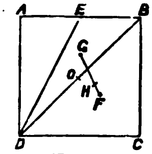


Fig. 3.

fulfil the requirements of case (4), it must have been placed to the right of the line DE (joining D and the mid-point of AB). Thus, in only $\frac{3}{4}$ of the possible positions is case (4) practicable, and in only $\frac{1}{2}$ is case (2). Now the average positions for the fulfilment of the several requirements are the mathematical centres of the spaces in which they are possible. For case (1) and case (3), the centre of the table is the average position, and the mean value of a and b is $\frac{1}{2}$ each. For case (2), the centre of the triangle is the average position, and within the said limits the mean value of a is $\frac{1}{3}$, of b $\frac{2}{3}$. For case (4), the centre of the trapezoid is the average position, and the mean value of a is $\frac{4}{9}$, of b, $\frac{11}{18}$. Under these circumstances, and since the second terms within brackets in the equations for (4) and (3) are governed by (2) and (4) respectively, the average value of $\frac{b-a}{b(1+a)}$, in the equation for case (4), in those instances in which the term counts at all, will be $\frac{\frac{2}{3}-\frac{1}{3}}{\frac{2}{3}(1+\frac{1}{3})} = \frac{3}{8}$; and of $\frac{2b-a}{2b+ab}$, in the equation for case (3), will be $\frac{\frac{22}{18}-\frac{8}{18}}{\frac{22}{18}+\frac{44}{162}} = \frac{63}{121}$. But we must multiply these by the factors denoting their proportionate occurrence, to find their mean values, namely, for case (4), $\frac{3}{8}\times\frac{1}{2} = \frac{3}{16}$; and, for case (3), $\frac{63}{121}\times\frac{3}{4} = \frac{189}{484}$.

Then the mean tangents of the compound angles will be as follows:— 1·000000, ·912500, ·609504, and of the angle for case (1), ·333333. The angles corresponding to these tangents are 45°, $42^\circ\,22'\,49\frac{3}{5}''$, $31^\circ\,21'\,45''$, and $18^\circ\,26'\,0\frac{3}{5}''$. Consequently, the mean angles for each separate case are, (1) $18^\circ\,26'\,0\frac{3}{5}''$, (2) $2^\circ\,37'\,10\frac{3}{5}''$, (3) $12^\circ\,55'\,44\frac{1}{5}''$, (4) $11^\circ\,1'\,4\frac{3}{5}''$. And these represent the following probabilities:—(1) ·4096, (2) ·0582, (3) ·2873, (4) ·2449.

7607. (By T. Muir, M.A., F.R.S.E.)—Prove that [see Quest. 7574]

$$\begin{vmatrix} a, & b & . & \dots & . & . \\ c, & a, & b & \dots & . & . \\ . & c, & a & \dots & . & . \\ & \dots & & \dots & & \dots \\ . & . & . & \dots & a, & b \\ . & . & . & \dots & c, & a \end{vmatrix}_n = \left(a-2(bc)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a-2(bc)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\dots\left(a-2(bc)^{\frac{1}{2}}\cos\frac{n\pi}{n+1}\right),$$

or what is equally general,

$$K(a^ba^b\dots{}^ba)_n = \left(a-2(-b)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\dots\left(a-2(-b)^{\frac{1}{2}}\cos\frac{n\pi}{n+1}\right).$$

Solution by B. Hanumanta Rau, M.A.; Sarah Marks; *and others.*

Let $a = (bc)^{\frac{1}{2}}x$, and let D_n represent the given determinant; then

$$D_n = aD_{n-1} - bcD_{n-2} \text{ or } D_n - (bc)^{\frac{1}{2}}xD_{n-1} + bcD_{n-2} = 0,$$

therefore D_n is the coefficient of y^n in the expansion of

$$[1-(bc)^{\frac{1}{2}}xy+bcy^2]^{-1} \text{ or } (bc)^{\frac{1}{2}n}\times\text{coefficient of } z^n \text{ in } (1-xz+z^2)^{-1},$$

$\therefore\ D_n = (bc)^{\frac{1}{2}n}\left(x-2\cos\frac{\pi}{n+1}\right)\left(x-2\cos\frac{2\pi}{n+1}\right)\ldots\left(x-2\cos\frac{n\pi}{n+1}\right)$

[see Solution of Question 7574]

$$= \left(a-2\,(bc)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a-2\,(bc)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\ldots.$$

Again, $K\,(a^b a^b \ldots {}^b a)_n = \begin{vmatrix} a, & b, & 0, & 0 & \ldots \\ -1, & a, & b, & 0 & \ldots \\ 0, & -1, & a, & b & \ldots \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix} = D_n$ where $c = -1$

$$= \left(a-2\,(-b)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a-2\,(-b)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\ldots.$$

7766. (By R. TUCKER, M.A.)—If ρ, ρ', ω are the "T. R." and Brocard radii, and Brocard angle respectively of a triangle, prove that (1) $\dfrac{\cos 3\omega}{\cos\omega} = \left(\dfrac{\rho'}{\rho}\right)^2$; and (2), if ρ_1, ρ_2 are the "T. R." radii in the ambiguous case of triangles, then $\rho_1\cos\omega_1 = \rho_2\cos\omega_2$.

Solution by the PROPOSER; B. HANUMANTA RAU, M.A.; *and others.*

Referring to equations (xii.) and (xviii.) of the article on the "Triplicate-Ratio Circle" (*Quarterly Journal of Mathematics*, Vol. xix., No. 76), we see that

$$\cos\omega = \frac{k}{2\lambda},\ \text{therefore}\ \frac{\cos 3\omega}{\cos\omega} = 4\cos^2\omega - 3 = \frac{k^2}{\lambda^2} - 3 = \left(\frac{\rho'}{\rho}\right)^2;$$

we also get at once $\rho'^2 + 3\rho^2 = R^2$, a result which is obtained geometrically by Dr. CASEY in his *Sequel to Euclid*, 3rd ed., p. 167. From the same article, we get $2\rho_1\cos\omega_1 = R$, $2\rho_2\cos\omega_2 = R'$. Hence in *all* cases when the circum-radii are equal, as they are in the ambiguous case,

$$\rho_1\cos\omega_1 = \rho_2\cos\omega_2.$$

7415. (By the Rev. T. C. SIMMONS, M.A.)—Two conics have a common focus, about which one of them is turned. Prove that the enveloping conic of the common chord depends only on the positions of the directrices, and the ratio of the eccentricities, of the original conics; and hence, when these are known, give an easy method of constructing it. [See Question 4417, *Reprint*, Vol. 39, p. 117.]

Solutions by (1) Dr. CURTIS; (2) *the* PROPOSER.

1. The conics are represented by the equations

$$r = e\,(x\cos\alpha + y\sin\alpha - p),\quad r = e'\,(x\cos\beta + y\sin\beta - p')\ldots\ldots\ldots(1,\,2),$$

where α, p, p' are given, and β is variable; hence their common chord and its envelope are

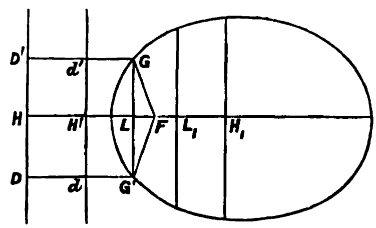
$$e'x\cos\beta + e'y\sin\beta - [e'p' + e(x\cos\alpha + y\sin\alpha - p)] = 0,$$

$$e'^2(x^2+y^2) = [e'p' + e(x\cos\alpha + y\sin\alpha - p)]^2,$$

or
$$r = \frac{e}{e'}\left\{x\cos\alpha + y\sin\alpha - \left(p - \frac{e'}{e}p'\right)\right\},$$

a conic confocal with the other two, whose eccentricity is e / e' and whose directrix is parallel to that of the fixed conic, at a distance from it $= e'p'/e$, and nearer to the focus.

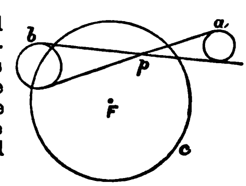
To construct this conic, let us suppose $\beta = \alpha$, then G, G′, the points of intersection of the conics, will lie on a line parallel to the common direction of DD′, dd', the directrices of the fixed and of the movable conic, respectively, cutting FH′H, the perpendicular from the common focus F on the directrices, in L, so that HL : H′L :: e' : e.



Again, suppose $\beta = \pi + \alpha$, then the directrix of the movable conic will be parallel to DD′ and at a distance FH_1 from F, where $FH_1 = FH'$; find L_1 so that $HL_1 : H_1L_1 :: e' : e$, then it is plain that L and L_1 will be the extremities of the axis major of the required conic, and, as one of its foci F is known, it is thus fully determined.

2. Dr. Curtis's solution, admirable in other respects, leaves out of sight the other chord of intersection, with respect to which, as is seen below, the theorem also holds. In fact, as r is really a surd quantity in x and y, equations (1) and (2) ought perhaps in strictness to be squared before subtracting.

Let A be the fixed, B the movable conic, and let them be reciprocated with respect to the common focus F into two circles a and b; then a is fixed and the centre of b revolves in a circle centre F; and, since the centre of b is the pole of the directrix of B, if we take this as the reciprocating circle, the directrix of B will always touch it.



Now, since p, the intersection of the common tangents of a and b, always divides the line joining the centres of a and b in the same ratio, the locus of p is a circle which in conjunction with C has the centre of a for one centre of similitude. That is to say, the common tangents of a and b intersect in a circle whose common tangents with the circle C intersect in the centre of a.

Hence in the original figure the chord of intersection of A and B touches a conic with the same focus, and whose intersection with the circular locus of the foot of the directrix of B lies on the directrix of A. We have, then, two points on the enveloping conic depending only on the positions of the directrices of A and B; and since, as shown above, its vertex depends only on the ratio of the eccentricities of A and B, and its focus is known, it is completely determined, and the required result follows. It will also be

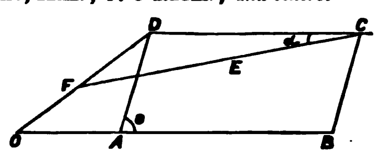
seen that the above reasoning applies equally to either of those two chords of intersection which in the initial position are perpendicular to the common axis.

If the directrix $dH'd'$ of the movable conic is further from the focus than DHD′ the fixed directrix, describe a circle centre F and radius FH′. The two points where it meets DHD′ will lie on both of the enveloping conics, a striking result, since the points depend only on the distances of the nearer directrices from the common focus, and are entirely independent of the nature of the original conics. It can be also seen analytically (but for one of the enveloping conics only) from Dr. Curtis's equation

$$r = \frac{e}{e'}\left\{x\cos\alpha + y\sin\alpha - \left(p - \frac{e'}{e}p'\right)\right\} \text{ by putting } x\cos\alpha + y\sin\alpha - p = 0.$$

7617. (By D. Biddle.)—Let a parallelogram ABCD have one side AB fixed and the other three capable of movement in one plane by hinge-action; and within the parallelogram let CE form a given angle with CD; then, if O be a fixed point in BA produced, and F, F′, &c. the points of intersection of CE, C′E′ with OD, O′D′, &c.; find the locus of F′.

Solution by B. Hanumanta Rau, M.A.; J. O'Regan; *and others.*



Let $OA = a$, $OB = b$, $AD = c$; then the equations to OD, CF are $\dfrac{y}{x} = \dfrac{c\sin\theta}{a + c\cos\theta}$,

$y - c\sin\theta = (x - b - c\cos\theta)\tan\alpha$,

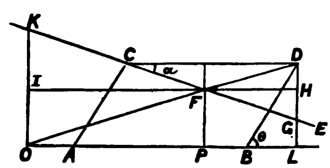
and, as the coordinates of O satisfy both these equations, we get

$$c\sin\theta = y + \frac{(a-b)\tan\alpha \,.\, y}{x\tan\alpha - y} \text{ and } c\cos\theta = x - a + \frac{(a-b)\tan\alpha \,.\, x}{x\tan\alpha - y},$$

$$\text{therefore } c^2 = \left\{y + \frac{(a-b)\tan\alpha \,.\, y}{x\tan\alpha - y}\right\}^2 + \left\{(x-a) + \frac{(a-b)\tan\alpha \,.\, x}{x\tan\alpha - y}\right\}^2,$$

the equation to a circular quartic.

[The Proposer remarks that, if C and D change places, the above reasoning will still apply, although one or two of the signs will need altering; thus, let $OB = a$, $OA = b$, $AC = c$, $OP = x$, $FP = y$; then,



$DL = c\sin\theta$; $BL = c\cos\theta$;

$IK = a\tan\alpha$; $DG = (a-b)\tan\alpha$.

But $\quad DH : DG = FP : IK + FP$,

$$\text{therefore } \quad \frac{DG \,.\, FP}{IK + FP} = DH, \quad DL = FP + \frac{DG \,.\, FP}{IK + FP};$$

i.e., $\quad c\sin\theta = y + \dfrac{(a-b)\tan\alpha \,.\, y}{x\tan\alpha + y}$; similarly $PL : OP = DG : IK + FP$;

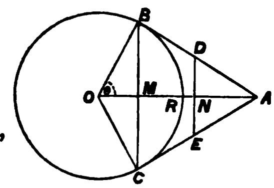
$$\text{therefore} \qquad \frac{\mathrm{DG}\,.\,\mathrm{OP}}{\mathrm{IK}+\mathrm{FP}} = \mathrm{PL}, \quad \text{and} \quad \mathrm{BL} = \frac{\mathrm{DG}\,.\,\mathrm{OP}}{\mathrm{IK}+\mathrm{FP}} - \mathrm{PB};$$

$$i.e., \qquad c\cos\theta = (x-a) + \frac{(a-b)\tan\alpha\,.\,x}{x\tan\alpha+y},$$

$$\text{and} \qquad c^2 = \left\{y + \frac{(a-b)\tan\alpha\,.\,y}{x\tan\alpha+y}\right\}^2 + \left\{(x-a) + \frac{(a-b)\tan\alpha\,.\,x}{x\tan\alpha+y}\right\}^2\Big].$$

7146. (By Professor GENESE, M.A.)—AB, AC are two tangents to a circle, and DE bisects them; show (1) that DE cannot meet the circle; and hence prove (2) that for acute angles $\theta - \sin\theta < \tan\theta - \theta$.

Solution by B. REYNOLDS, M.A.; Professor NASH, M.A.; *and others.*



$$\mathrm{ON} = \mathrm{OM} + \tfrac{1}{2}\mathrm{MA} = r\cos\theta + \tfrac{1}{2}\mathrm{BA}\sin\theta$$
$$= r(\cos\theta + \tfrac{1}{2}\tan\theta\sin\theta)$$
$$= r\,\frac{2\cos^2\theta + \sin^2\theta}{2\cos\theta},$$
$$= r\,\frac{1+\cos^2\theta}{2\cos\theta} = r\,\frac{(1-\cos\theta)^2 + 2\cos\theta}{2\cos\theta},$$
$$= r\left\{1 + \frac{(1-\cos\theta)^2}{2\cos\theta}\right\},$$

and is thus greater than r, θ being always acute. Hence DNE cannot cut the circle.

Again, BD + DE + EC > arc BRC, or BA + BM > arc BRC,

or $\qquad r\tan\theta + r\sin\theta > 2r\theta$, whence $\theta - \sin\theta < \tan\theta - \theta$.

[DE is the radical axis of the circle and a point-circle at A, and cannot therefore meet the circle.]

7752. (By ASPARAGUS.)—From a point on one of the common chords perpendicular to the transverse axis of two confocal conics are drawn tangents OP, OQ, OP′, OQ′ to the two conics: prove that the straight lines PP′, PQ′, P′Q, P′Q′ each pass through one of the common foci.

Solution by Dr. CURTIS; Professor NASH, M.A.; *and others.*

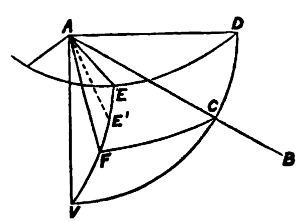
It is plain that, if through the intersection of a centre of similitude of two non-intersecting circles a straight line be drawn intersecting them, the tangents at each pair of corresponding points are parallel; and, if we reciprocate this theorem with regard to any circle whose centre is at one of the limiting points of the two circles, we obtain the proposed theorem.

[A *direct* proof of this theorem is not difficult, and may readily be found.]

7688 & 7761. (By W. J. C. SHARP, M.A.)—(7688.) Find the curve in which a kite-string will hang, when acted on by a uniform wind blowing in a direction inclined to the horizon.

(7761.) A flexible string is suspended slackly from two fixed points, and acted upon by a uniform horizontal wind, blowing in a direction making any angle with the horizontal projection of the line joining the points. Find the curve in which the string hangs and the tension at any point.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.



Let A, B be the fixed points; with A as centre, let a sphere of radius unity be drawn, ADCV being the vertical great circle whose plane contains AB, and AVE the great circle whose plane is parallel to AE, the direction of the wind, supposed horizontal, let f denote the accelerating force of the wind on the element and g that of gravity; the combined action of the two will be in the direction AF, where tan VF $= f/g$, the great circle CF determines the plane in which the string will rest, and, as each equal element is acted on by a constant force in the invariable direction, AF, the curve of equilibrium will be a catenary, the same as would result if the plane AFB were turned round AB into a vertical position, and then turned along with AB round its axis until AF were vertical; all the known relations for the catenary as to tension, &c., hold, g in the usual formulæ being replaced by $h \equiv (f^2+g^2)^{\frac{1}{2}}$.

If the known direction of the wind be not supposed to be horizontal, let AE′ denote its direction, and AF will be determined by the condition $\dfrac{\sin \mathrm{VF}}{\sin \mathrm{FE'}} = \dfrac{f}{g}$, and $h \equiv (f^2+g^2+2fg\cos \mathrm{VE'})^{\frac{1}{2}}$, all else being as before.

7744. (By Professor COCHEZ.)—Parmi les courbes isoperimètres planes passant par deux points fixes, quelle est celle qui par sa révolution autour de l'axe des x, engendre la surface maximum ou minimum?

Solution by DR. CURTIS; Professor MATZ, M.A.; *and others.*

As, by GULDIN's theorem, the area of the generated surface is equal to the length of the curve multiplied by the circumference of the circle, or known proportional part of the circumference of the circle, described by the centre of gravity of the curve, this area will be maximum, or minimum, when the centre of gravity of the curve is at the greatest, or least, distance from the axis of X, that is, in the forms which would be the positions of equilibrium assumed by the string if each element were acted on by a constant force perpendicular to X, *from* it for maximum, and *towards* it for minimum, viz., a catenary concave towards X in the first case, and convex in the second.

7836. (By Professor SYLVESTER, F.R.S.)—If p, q be two matrices (to fix the ideas, suppose of the third order), which have one latent root in common, and let λ', λ''; μ', μ'' be the other latent roots of p, q; prove that the product $(p-\lambda')(p-\lambda'')\,X\,(q-\mu')(q-\mu'')$ (where X is an arbitrary matrix) is of invariable *form*, the only effect of the intermediate arbitrary matrix being to alter the value of each term of the product in a constant ratio; *i.e.*, in the nomenclature of the New Algebra,

$$(p-\lambda')(p-\lambda'')\,X\,(q-\mu')(q-\mu'')$$

is constant to a *scalar* multiple *près*.

For the benefit of the learner, I recall that if $p = \begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$ the roots of the algebraical equation $\begin{vmatrix} a-\lambda & b & c \\ a' & b'-\lambda & c' \\ a'' & b'' & c''-\lambda \end{vmatrix} = 0$ are called the *latent roots* of p, the equation itself being called the latent equation, and the function equated to 2-en the latent function.

Solution by the PROPOSER.

To show that, if p, q (two ternary matrices) have the latent roots $\lambda = \mu$ in common, and λ', λ''; μ', μ'' are the other two latent roots of p, q respectively, $(p-\lambda')(p-\lambda'')\,X\,(q-\mu')(q-\mu'')$ is constant to a scalar factor *près*. Let u be the matrix above written, then $(p-\lambda)\,u = 0$, $u\,(q-\mu) = 0$, hence $pu-uq = 0$. If, then, we write out the constituents of p, q, and of u, we shall obtain from the above equation 9 homogeneous linear equations for determining the last-named 9, and the resultant of these equations will be identical with the resultant of the latent functions of p and q; the ratios of the 9 constituents of u will be determined from selecting any 8 out of the 9 equations which are perfectly independent of X; hence u is known to a scalar factor *près* in terms of the elements of p and q.

Of course, the same reasoning applies to matrices of any order ω; the corresponding u, it should be noticed, will always have $\omega-1$ degrees of nullity, *i.e.*, all the minor determinants to u of the second order will be equal to zero. We may write $u = c\,(p-\lambda')(p-\lambda'')(q-\mu')(q-\mu'')$, and obtain an identical equation to u, of which the coefficients will be rational integer functions of the coefficients to the identical equations to p and q respectively.

7838. (By the late Professor CLIFFORD, F.R.S.)—Prove that a string will rest in the form of a circle if it be repelled from a point in the circumference with a force inversely as the cube of the distance.

Solution by ÂSÛTOSH MUKHOPÂDHYÂY.

Suppose that the string has assumed the form of a circle of radius a, under the action of a repulsive central force at any point O on the circumference; and, since the osculating plane at every point contains the centre of force, and since two consecutive osculating planes have a tangent line to the string common, it easily follows that the string lies wholly

in one plane. Let r be the radius vector from the origin O to any point P, θ being its inclination to the diameter OA, which is the initial line; and, if p be the perpendicular from O on the tangent at P, we have $p = r\cos\theta$, and $r = 2a\cos\theta$, which gives $r^2 = 2ap$; or, taking u as the reciprocal of the radius vector, we have $u^2 = (2ap)^{-1}$, and

$$du = d(p^{-1})/4au \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, if T and $T+dT$ be the tensions at the extremities of an element of the string at P, we have, as usual, $Tp = h$ ....(2), and, if F be the force, and m the element of mass of the string, we have

$$dT = -mFdr \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

(See Professor TOWNSEND's Classical Paper "On the Analogy between the Curve of Free Equilibrium of a String under a Central *Repulsive* Force, and the Free Orbit of a Particle under a Central *Attractive* Force," *Quarterly Journal of Mathematics*, Vol. xiii., p. 217.)

Eliminating T between (2) and (3), we have

$$F\,du = \frac{h}{m}u^2 d\left(\frac{1}{p}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating du between (1) and (4), we see that $F \propto u^3$.

According to the analogy pointed out in Professor TOWNSEND's paper, it follows that the law of force under which a particle would freely describe the same circle is that of the inverse *fifth* power of the distance. (See NEWTON's *Principia*, Lib. 1, Prop. 7.)

The inverse problem—viz., given the law of force, to determine the orbit—is also easily solved. In fact, from what has been said above, joined with the fact that $r^4 = p^2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}$, it follows at once that the orbit is $d\theta = \frac{C\,dr}{r(T^2r^2 - C^2)^{\frac{1}{2}}}$, where C is a constant. Hence it follows that, if the force varies inversely as the μth power of the distance, the tension is equal to $\frac{mdf}{\mu-1}\cdot\frac{1}{r^{\mu-1}}$, where d is a constant, and f the magnitude of the force exerted at unit distance on the unit mass of matter. Substituting for T, we get the general equation to the curve in the form

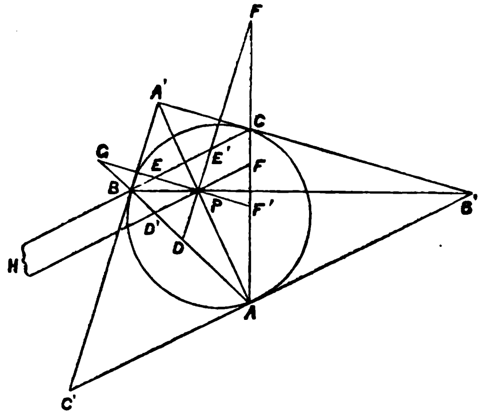
$$\left(\frac{d}{r}\right)^{\mu-2} = \cos(\mu-2)\,\theta.$$

If $\mu = 3$, we get $r = d\cos\theta$, which denotes a circle, of diameter d, the origin being on the circumference, which is Professor CLIFFORD's theorem.

7612. (By W. S. MCCAY, M.A.)—If O be the centre of perspective of the triangle ABC, and the triangle formed by the tangents at the vertices to the circumcircle; and if through O parallels be drawn to the tangents cutting the sides internally in six points, and externally in three points; prove that (1) the six internal points lie on a circle whose centre is P and radius $abc/(a^2+b^2+c^2)$; (2) these points are vertices of two equal triangles, similar to ABC; (3) these points are vertices of three rectangles inscribed in ABC having a common circum-circle; (4) the three external intersections of the sides with the lines through P are collinear.

I. *Solution by* HANUMANTA RAU, M.A.; G. G. STORR, B.A.; *and others.*

Let A′B′C′ be the triangle formed by the tangents. Then P is the point of intersection of AA′, BB′, CC′. Taking ABC as the triangle of



reference, the equations to B′C′, C′A′, A′B′; AA′, BB′, CC′ are

$$\frac{\beta}{b}+\frac{\gamma}{c}=0,\quad \frac{\gamma}{c}+\frac{\alpha}{a}=0,\quad \text{and}\quad \frac{\alpha}{a}+\frac{\beta}{b}=0.$$

$$\frac{\gamma}{c}-\frac{\beta}{b}=0,\quad \frac{\gamma}{c}-\frac{\alpha}{a}=0,\quad \frac{\alpha}{a}-\frac{\beta}{b}=0,$$

hence the coordinates of P are given by

$$\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}=\frac{2\Delta}{a^2+b^2+c^2},$$

and P is thus seen to be the *point de Grebe* of the triangle.

If DEF be drawn through P parallel to A′C′, then

$$\text{PD}=\frac{\gamma}{\sin \text{PDB}}=\frac{\gamma}{\sin \text{C}}=\frac{2\Delta\,.\,\text{C}}{\sin \text{C}\,(a^2+b^2+c^2)}=\frac{abc}{a^2+b^2+c^2}=\text{PE}.$$

Similarly, the other internal points of intersection are at the same distance from P; also $\angle\,\text{EDF}=\frac{1}{2}\text{EPF}=\frac{1}{2}(\pi-\text{B}')=\text{CBA}$,

$$\angle\,\text{D}'\text{E}'\text{F}'=\tfrac{1}{2}\text{D}'\text{PF}'=\tfrac{1}{2}(\pi-\text{B}')=\text{CBA},$$

and so on; hence the triangles EDF, D′E′F′ are similar to ABC, and, since they are inscribed in the same circle, they are equal.

This is obvious from the fact that any two diameters of a circle determine an inscribed rectangle. [See Question 7747 and its solution.]

The equation to DE′F, which passes through P (a, b, c) and is parallel to C′A′ $\left(\frac{\alpha}{a}+\frac{\gamma}{c}=0\right)$, is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ a, & b, & c \\ \frac{b}{c}, & \frac{c}{a}-\frac{a}{c}, & -\frac{b}{a} \end{vmatrix}=0.$$

Putting $\beta = 0$, the equation to BF becomes

$$\left(a - \frac{b^2}{a} - \frac{c^2}{a}\right)\alpha + \left(c - \frac{a^2}{c} - \frac{b^2}{c}\right)\gamma = 0,$$

or $$\frac{2bc\cos A}{a}\alpha + \frac{2ab\cos C}{c}\gamma = 0, \quad \text{or} \quad \frac{\alpha\cos A}{a^2} + \frac{\gamma\cos C}{c^2} = 0.$$

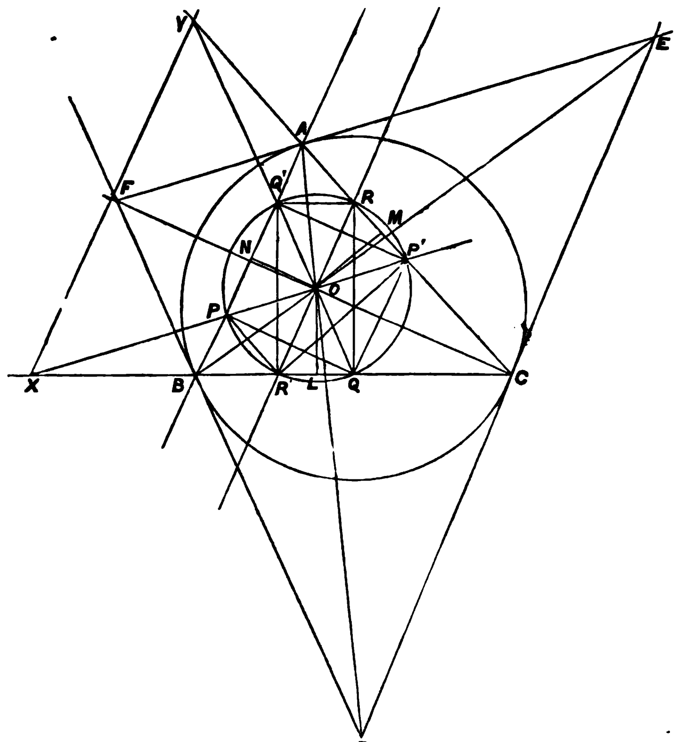
Similarly the lines CG and AH are represented by

$$\frac{\alpha\cos A}{a^2} + \frac{\beta\cos B}{b^2} = 0, \quad \text{and} \quad \frac{\beta\cos B}{b^2} + \frac{\gamma\cos C}{c^2} = 0.$$

The points F, G, H therefore lie on the straight line

$$\frac{\alpha\cos A}{a^2} + \frac{\beta\cos B}{b^2} + \frac{\gamma\cos C}{c^2} = 0.$$

II. *Solution by* J. Brill, B.A.; Nilkanta Sarkar, M.A.; *and others.*



1. Draw OL, OM, ON perpendicular to the sides; then
$\angle ABD = 180° - ABF = 180° - C$, similarly $\angle ACD = 180° - B$;

$\therefore\ \dfrac{AD}{BD} = \dfrac{\sin ABD}{\sin BAD} = \dfrac{\sin C}{\sin BAD}$; also $\dfrac{AD}{CD} = \dfrac{\sin ACD}{\sin CAD} = \dfrac{\sin B}{\sin CAD}$.

Now BD = CD, therefore $\dfrac{\sin BAD}{\sin CAD} = \dfrac{\sin C}{\sin B}$,

But OM = OA sin CAD, and ON = OA sin BAD,

therefore $\dfrac{OM}{ON} = \dfrac{\sin CAD}{\sin BAD} = \dfrac{\sin B}{\sin C} = \dfrac{b}{c}$; similarly $\dfrac{OL}{OM} = \dfrac{a}{b}$,

therefore $\dfrac{OL}{a} = \dfrac{OM}{b} = \dfrac{ON}{c} = \dfrac{a.OL + b.OM + c.ON}{a^2+b^2+c^2} = \dfrac{2\Delta}{a^2+b^2+c^2}$.

Hence we have $\qquad OL = \dfrac{2a\Delta}{a^2+b^2+c^2} = \dfrac{abc \sin A}{a^2+b^2+c^2}$.

Now $\angle OQL = \angle CBD = A$, therefore $OQ \sin A = OL = \dfrac{abc \sin A}{a^2+b^2+c^2}$,

$\therefore\ OQ = \dfrac{abc}{a^2+b^2+c^2} = OQ' = OP = OP' = OR = OR'$ (from symmetry).

2. We have proved that P, Q, R, P′, Q′, R′ all lie on a circle having its centre at O. Now PP′ is a diameter of this circle, therefore ∠ PRP′ is a right angle. Similarly it may be proved that the angles QPQ′ and RQR′ are right angles, therefore the sides of the triangle PQR are respectively perpendicular to those of the triangle ABC, therefore the two triangles are similar.

In like manner it can be proved that the triangle P′Q′R′ is also similar to the triangle ABC.

Moreover OP = OP′ and OQ = OQ′, also ∠ POQ = ∠ P′OQ′, therefore PQ = P′Q′; similarly it may be proved that QR = Q′R′, and RP = R′P′. Thus we have two equal triangles PQR and P′Q′R′, each of them similar to the triangle ABC.

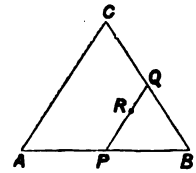
3. Since POP′ and QOQ′ bisect each other, they are the diagonals of a parallelogram, therefore PQP′Q′ is a parallelogram. But the angles PQP′, QP′Q′, P′Q′P, Q′PQ are right angles, being angles in semicircles, therefore PQP′Q′ is a rectangle. Similarly, QRQ′R′ and RPR′P′ are rectangles, therefore the points P, Q, R, P′, Q′, R′ are the vertices of three rectangles inscribed in ABC, and having a common circumcircle.

4. We have $\qquad \angle BPX = \angle APP' = \angle PAF = C$;
therefore $\angle BPP' + \angle BCP' = \angle BPP' + \angle BPX = 2$ right angles;
therefore circle passes round BPP′C; therefore BX.CX = PX.P′X, Therefore X is a point on the radical axis of the two circles. Similarly it may be shown that Y and Z lie on the radical axis; *i.e.*, X, Y, Z are collinear.

[The circle PQRP′Q′R′ makes intercepts on the sides proportional to the cosines of the angles, whence Professor Casey (in the 3rd edition of his *Sequel to Euclid*) proposes to call this circle the "Cosine-Circle."]

7849. (By the Rev. T. C. Simmons, M.A.)—If from a random point within an equilateral triangle perpendiculars are drawn on the sides, show that the respective chances that they can form (1) *any* triangle, (2) an acute-angled triangle, are $p_1 = \frac{1}{4}$, $p_2 = 3 \log_e 2 - 2 = \cdot 07944 = \frac{5}{63}$ nearly.

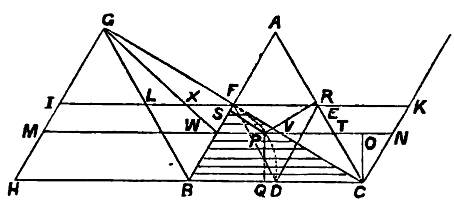
Solutions by (1) H. McColl, B.A.; (2) D. Biddle.



1. Let ABC be the equilateral triangle. In AB take any random point P. Draw PQ parallel to AC, and meeting BC at Q. In PQ take a random point R. Then R may be taken as the random point spoken of. Taking AB as unity, let $x = \text{AP}$, and let $y = \text{PR}$. The perpendiculars from R upon the sides are proportional to the algebraic quantities x, y, $1-x-y$. The chance that *no* triangle can be formed by the perpendiculars is therefore 3 times the chance that $1-x-y$ will be greater than $x+y$, that is to say, 3 times the chance that $y-\frac{1}{2}+x$ will be negative, all values of x between 0 and 1, and of y between 0 and $1-x$, being equally probable. The chance that $y-\frac{1}{2}+x$ will be negative is a fraction the numerator of which is the value of the integral $\int dx \int dy$ when the limits for x are 0 and $\frac{1}{2}$, and those for y are 0 and $\frac{1}{2}-x$, and the denominator of which is the value of the same integral when the limits of x are 0 and 1, and those of y are 0 and $1-x$. The chance of *no* triangle is therefore $\frac{3}{4}$, and of *some* triangle $\frac{1}{4}$.

The chance that *no acute-angled triangle* can be formed by the perpendiculars is 3 times the chance that $(1-x-y)^2$ will be greater than x^2+y^2, that is to say, 3 times the chance that $y-\dfrac{1-2x}{2-2x}$ will be negative. The chance that this will be negative is a fraction, the denominator of which will be the same as in the former case, but the numerator of which will be the integral $\int dx \int dy$, between the limits 0 and $\frac{1}{2}$ for x, and 0 and $\dfrac{1-2x}{2-2x}$ for y. The chance of *no acute-angled triangle* is therefore $3(1-\log 2)$; and the chance of an acute-angled triangle is $1-3(1-\log 2)$, which $= 3\log 2 - 2$. [This value, in decimals, is ·07944154. See the solution of Question 3342, *Reprint*, Vol. xv., p. 58, and Vol. xvi., p. 68.]

2. *Otherwise*:—To meet the first-named conditions, it is evident the point must lie within the space bounded by the lines joining the mid-



points of the sides of the triangles, since these lines form the locus of equality between one perpendicular and the sum of the other two. The area of the said space is one-fourth of the whole triangle. Hence $p_1 = \frac{1}{4}$.

To find p_2 we must discover the locus of points from which perpendiculars may be drawn so that the sum of the squares of two shall equal the square of the third; in other words, the locus for the formation of right-angled triangles, which are the limit of acute. The mid-points of the sides are, as before, the starting points, but the course consists of three curves instead of straight lines. Let ABC be the triangle, D, E, F the

mid-points of its sides, and P a point placed so that $PQ^2+PS^2=PR^2$. It is easily seen that $PS:PW=PR:PT=PQ\,(=CO):CT\,(=NT)$. But $PW+PT+NT=WN=BC=1$. Consequently the problem resolves itself into dividing a given line into three portions, so as to form a right-angled triangle, or rather a series of right-angled triangles with one side increasing arithmetically from 0 to $\frac{1}{2}$ (represented by NT, between 0, at C, and $\frac{1}{2}$, at RK). Now $WP^2=PT^2-NT^2$, and $PT=1-WP-NT$, therefore $WP=\dfrac{1-2NT}{2-2NT}$. If, then, we produce CB to H, and on BH describe BGH = ABC; and if we join GC, cutting AB in its mid-point F, and draw KI through F parallel to CH, we are able to find P for any given height (less than half that of the triangle) above BC. Thus, by drawing NM at the given height parallel to CH, and drawing FP parallel to GW, we cut NM in the point required. For, $NV=2NT$, $MV=2-2NT$, $WV=1-2NT$, and $IF=1$. Therefore

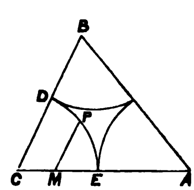
$$XF\,(=WP):WV\,(=1-2NT)=IF\,(=1):MV\,(=2-2NT).$$

The curve DPF is indicated by a series of similar intersections. To find the area of the space cut off by it, we must find the mean length of the lines represented by WP, when NT varies from 0 to $\frac{1}{2}$. Let $NT=r$, then $\dfrac{1-2r}{2-2r}=\frac{1}{2}\,(1-r-r^2-r^3-r^4-\ldots-r^\infty)$, which, by giving the mean values of the several powers of r, when r varies from 0 to $\frac{1}{2}$, becomes

$$\tfrac{1}{2}\left(1-\frac{1}{2\,.\,2}-\frac{1}{4\,.\,3}-\frac{1}{8\,.\,4}-\frac{1}{16\,.\,5}-\&c.\right),$$

or, to six places of decimals, $\frac{1}{2}\,(1-{\cdot}386294)={\cdot}306853$, which is also the proportionate area of the space as compared with the whole triangle. The proportionate area of the three spaces $={\cdot}920558$, leaving ${\cdot}079442$ for the space within which P must lie in order that acute-angled triangles may be formed according to clause (2). Hence $p_2={\cdot}07944$, or about $\frac{2}{25}$.

[Mr. Biddle's solution, effected as it is without the use of either Analytical Geometry or the Integral Calculus, is of much interest. The following solution, by their aid, while regarding the question from the same point of view, has been sent by Mr. Simmons, as being perhaps somewhat simpler :—



After proving as above, that $p_1=\frac{1}{4}$, take any point P on the curve $\alpha^2+\beta^2=\gamma^2$, and let PM drawn parallel to BC $=y$, CM $=x$. Then $\alpha=x\sin 60^\circ$, $y=\beta\sin 60^\circ$. Whence, since

$$\alpha^2+\beta^2=(a\sin 60^\circ-\alpha-\beta)^2,$$

we have $\quad x^2+y^2=(a-x-y)$ or $y=\dfrac{a^2-2ax}{2\,(a-x)}$,

so that the area of CDPE is

$$\frac{a}{2}\sin 60^\circ\int_0^{\frac{1}{2}a}\frac{a-2x}{a-x}\,dx=\frac{\sqrt{3}a}{4}\,(a-a\log 2)=\Delta\,(1-\log 2)\,;$$

and the whole space within ABC for which an acute-angled triangle is impossible is $3\Delta\,(1-\log 2)$; hence the chance of an acute-angled triangle is $\quad [\Delta-3\Delta\,(1-\log 2)]\,/\,\Delta$ or $3\log 2-2$.

Mr. Simmons states that he was not aware, before seeing Mr. McColl's solution, that the question had been previously proposed by him as Question 3342.]

7326. (By Professor HUDSON, M.A.)—Prove that, in the curve $r = a + b\cos\theta$, the polar subtangent cannot have a maximum or minimum value for a finite value of r, unless $a > b$.

Solution by B. HANUMANTA RAU, M.A.; J. O'REGAN; *and others.*

$$r = a + b\cos\theta,\quad \text{polar subtangent} = u = r^2\frac{d\theta}{dr} = -\frac{(a+b\cos\theta)^2}{b\sin\theta},$$

and, for a maximum or minimum value of u,

$$\frac{du}{d\theta} = 0,\quad \text{therefore}\quad 2(a+b\cos\theta) + \frac{(a+b\cos\theta)^2\cos\theta}{b\sin^2\theta} = 0,$$

or $2b(1-\cos^2\theta) + (a+b\cos\theta)\cos\theta = 0$, therefore $\cos\theta = \dfrac{a \pm (a^2+8b^2)^{\frac{1}{2}}}{4b}$.

In order that θ may be possible, we must have

$$4b \nless a \pm (a^2+8b^2)^{\frac{1}{2}},\quad \text{or}\quad 16b^2 - 8ab + a^2 \nless a^2 + 8b^2,\quad \text{therefore}\quad b \nless a.$$

4038. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—Prove that (1) a triangle can be partitioned into 13 triangles in 457 ways, two ways being reckoned the same if one is in any position the reflected image of the other, the size of the partitions being of no consequence; and find (2) in how many ways an equilateral triangle can be partitioned into 13 triangles of equal area.

Solution by the PROPOSER.

1. The number of partitions is the number of different triangles symmetric and asymmetric, which are found upon the 34 symmetrical and the 16 asymmetrical 9-acral 14-edra. These triangles are given by their edges, which are rapidly enumerated in the processes of my Theory of the Polyedra, nearly 25 years old.

There are 8 zoned polar edges (33), 7 zoneless polars, 30 epizonals, and 64 zonals; from which, and their zonal signatures, we find that there are 4 zoned polar, and 64 monozone, in all 68 symmetric, triangles on 34 symmetrical solids zoned and zoneless, namely, on 9 zoned polar, 7 zoneless polar, and 18 monozone solids. And on these 34 symmetric solids are 244 asymmetric edges, and 165 asymmetric triangles. This is evident from the signatures of symmetry, before the construction of a single figure.

The entire number of asymmetric edges on all the 9-acral 14-edra is entered, after a glance of inspection at the table preceding, thus, in two words, $\quad (33)\,512 = {}_0344^{4..} + 236_1 = 580,$

where 344 is the number of asymmetric diagonals that can be drawn in the 4-gons of the 9-acral 13-edra, and the 236 are asymmetric edges (33), across all of which lies a triangular section along an effaceable. Subtracting from these 580 the asymmetric edges found in the 34 symmetric solids, we get $580 - 244 = 336 = 21.16$; that is, there are 16 asymmetric solids. Every zonal and every zoneless polar edge is in one asymmetric

triangle, and every asymmetric edge is taken to be in two; and $2.580+64+7=1231$ is our number of these edges, which is to be diminished by one asymmetric edge for every monozone triangle. This gives $1231-64=1167=3.389$, proving that there are 389 asymmetric triangles. The sum of these and the 68 symmetric ones gives us 457 as the complete number of different triangles upon the 50 9-acral 14-edra. All this, and much more, is deduced and registered by the method of the Theory, before any attempt at construction by lines.

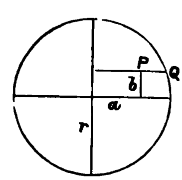
On any one p of the triangles upon any given one P of the 50 solids, the remaining 13 faces of P can be projected. The 13 triangles may by their shapes and unequal areas be varied in ways innumerable, while all the different figures will alike be the projection of the same P on this one face of it, p, taken for base. And can there be a reason why the 13 partitions should not all be of equal area, whether p be equilateral or not; so that 457 is the correct answer to both (1) and (2) in the question?

2. I cannot find nor recall my solution of (2), but I believe the answer to be $457-N$, where N is the number of different triangles on those 9-acral 14-edra which have fewer than two triaces. See a simpler question on this matter, which, I hope, will soon appear, and deserve discussion.

[That the preceding will be quite intelligible to our readers, Mr. KIRKMAN cannot flatter himself, limited as he is here in space. A clearer view of the subject may be obtained from a Memoir in the Proceedings of the Literary and Philosophical Society of Liverpool, Vol. XXXII., 1877–78, of which Memoir the title is "*The Enumeration and Construction of the 9-acral 9-edra.*"]

5218. (By the EDITOR.)—A circular target revolves uniformly around a vertical axis, lying in its plane and passing through its centre; and a shot is fired at the target in (1) a given or (2) a random direction: find, in the first case, the chance that the shot will hit the target, and show therefrom that, in the second case, the chance is $2/\pi$.

Solution by D. BIDDLE.



1. Let P be the point which the bullet would hit if the target were stationary and fully fronting the marksman, and a, b its coordinates. Then it is evident that, as the target revolves, the bullet, if it strikes the face at all, must hit it in the line PQ, and therefore during that portion of the revolution which, if r be the radius of the target, is represented by $2\cos^{-1}\left(\dfrac{a}{(r^2-b^2)^{\frac{1}{2}}}\right)$; and, if hitting the target back or front be allowed, then the angle is doubled, and this, divided by 2π, gives $\dfrac{2}{\pi}\cos^{-1}\left(\dfrac{a}{(r^2-b^2)^{\frac{1}{2}}}\right)=p_1$.

2. If an infinite number of horizontal lines be drawn on the target, a random shot may hit any one of those lines, and any point on that particular

line, supposing the target to be stationary as before. Consequently, when revolution takes place, the probability that the bullet will strike is the proportion borne by the mean apparent area of the target to the actual area, and this is the ratio between mean cosine and radius.

6960. (By Dr. MacAlister.)—Show from first principles that, if in any motion of a particle the tangential force be measured by the rate per second at which momentum is *increased*, the normal force will in the same units be measured by the rate per second at which momentum is *deflected*.

Solution by Âsûtosh Mukhopâdhyây.

Consider the motion of a particle of given mass under the action of given forces; then, the motion is completely defined as soon as we know two essentially distinct circumstances about it—viz., the length of the path described, and the curvature of the path at any point; that is to say, as soon as we know the equation connecting the intrinsic elements, arc (s) and the angle of deflection (ϕ) measured from any arbitrary but defined origins.

Regarding a curve as the limit of a polygon, it follows, from the definition of a tangent as the line forming two consecutive points, that the *length* of the arc depends on the tangential force, while its *form*, that is, its curvature, depends on the normal force; and, as the normal force is independent of the tangential force and *vice versâ*, it follows at once that, if we *define* the tangential force as the time variation of the *magnitude* ("increase") of the momentum, the other force in the system independent of this one, viz., the normal force, must be measured by its effect, that is, the normal force is the time variation of the *position* ("deflection") of the momentum. This is identically the same as the fact that the *length* and the *form* (curvature) are the two distinct and independent elements which are necessary and sufficient to define the path. It follows that this is the statement of the analytical theorem that, if the equation $M\frac{d^2s}{dt^2} = \frac{d}{dt}(Mv)$, is identically true, we have

$$M\frac{v^2}{\rho} = M\left(\frac{ds}{dt}\right)^2\frac{d\phi}{ds} = (Mv)\frac{d\phi}{dt},$$

where the equation to the curve $s = f(\phi)$ does not involve any extrinsic elements, but only the intrinsic elements s, ϕ.

[The Proposer remarks that he does not understand what is meant by the "time variation of the *position* of the momentum." If the *position* of the momentum at any moment increases the *position* of the particle, he demurs to the proposition; if it does not mean that, then a little explanation is wanted. Again, if the reasoning is so clear and direct as it appears in the above, ought it not to be possible to derive $\frac{Mv^2}{\rho}$ from first principles a little more inevitably?]

7803. (By the EDITOR.)—Trace the curve $y^2(x-a) = x^3-b^3$.

Solution by the REV. T. C. SIMMONS, M.A.; N. SARKAR, M.A.; *and others.*

The asymptotes are $\pm y = x+\frac{1}{2}a$ and $x = a$. Developing further, we see that, when x is positive, the curve lies above the first asymptote, and below when x is negative, also that it meets it when $x = \dfrac{4b^3-a^3}{3a^2}$. Now

$$\frac{dy}{dx} = \frac{2x^3-3ax^2+b^3}{2y(x-a)^2} = \infty \text{ at } (b,\ 0) \text{ and at } (a,\ 0),$$

also
$$= \pm\left(\frac{b^3}{4a^3}\right)^{\frac{1}{2}} \text{ at } \left\{0,\ \pm\left(\frac{b^3}{a}\right)^{\frac{1}{2}}\right\}.$$

At $(b,\ 0)$ the curve is of the form $\eta^2 = \dfrac{3b^2}{b-a}\xi$. Moreover, it is symmetrical with respect to Ox, and when a and b are unequal there is no real value of x intermediate to a and b.

The two figures are for (i.) $b>a$, (ii.) b positive and $<a$. When $b=a$, the curve becomes the line $x = a$ and a rectangular hyperbola. When $b = 0$, there is a cusp at the origin.

Fig. (i.)

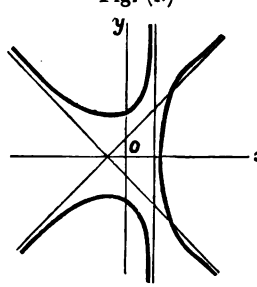
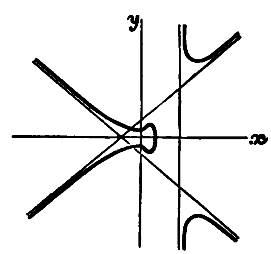


Fig. (ii.)



7797. (By D. EDWARDES.)—If

$$V_n = \int_0^1 [\log(1+x)]^n\,dx, \quad \text{prove that } V_n + nV_{n-1} = 2(\log_e 2)^n.$$

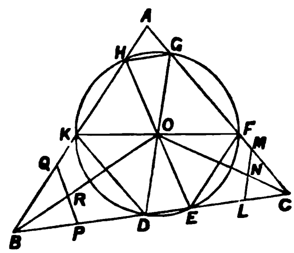
Solution by G. G. STORR, B.A.; W. T. MITCHELL, M.A.; *and others.*

Put $1+x = y$, then $V_n = \int_1^2 (\log y)^n\,dy$, hence we have

$$V_n = \left\{(\log y)^n y\right\}_1^2 - \int_1^2 n(\log y)^{n-1}\,dy = 2(\log 2)^n - nV_{n-1}.$$

7747. (By the EDITOR.)—Show that (1) in a triangle there can be inscribed three rectangles having each a side on one of the sides of the triangle, and their diagonals equal and crossing at their mid-points; and (2), if a, b, c be the sides of the triangle, the length of these equal diagonals is $2abc / (a^2+b^2+c^2)$.

Solutions by (1) J. McDOWELL, M.A.; (2) A. H. CURTIS, LL.D., D.Sc.

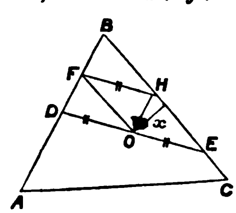


1. Let ABC be any triangle. Draw any straight line LM cutting BC, CA in L, M, and making $\angle$ CLM = A, and therefore CML = B, and any straight line PQ making $\angle$ BPQ = A, and therefore BQP = C. Let N be mid-point of LM, and R the mid-point of PQ, and let CN, BR meet in O. Through O draw DG parallel to LM and EH parallel to PQ, therefore the angles ODE and OED each = A; therefore DOE and GOH are identically equal isosceles triangles, and therefore HG is parallel to BC. Let the circle with centre O and radius OD, which, of course, passes through D, E, G, H, meet AB and AC again in K and F; then $\angle$ OKH = OHK = C, and OFG = OGF = B, hence therefore KOF is a straight line. Hence the three rectangles with their centres at O and equal diagonals are DEGH, FGKD, HKEF.

2. Since HG is parallel to BC, &c., let HG = la, DK = mb, EF = nc; $\therefore$ AH = lc, HK = nc, KB = mc; $\therefore$ $lc+nc+mc$ = AH + HK + KB = c; therefore $l+m+n=1$; hence, putting x for a diagonal, we have $\angle$ OFE = OEF = C, therefore $\frac{1}{2}$EF or $\frac{1}{2}nc = \frac{1}{2}x\cos C$; $\therefore n = x\cos C / c$; similarly $l = x\cos A / a$, $m = x\cos B / b$;

$$\therefore \quad x\left(\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}\right) = l+m+n = 1; \quad \therefore x = \frac{2abc}{a^2+b^2+c^2}.$$

Otherwise.—The locus of the mid-point of the diagonal of all rectangles inscribed in a triangle, and having a side along a, is the line joining the mid-point of a to the mid-point of the perpendicular from A on a. This locus and the two corresponding ones for the other two sides meet in a point (Quest. 7644) the trilinear coordinates of which are proportional to sin A, sin B, sin C, or are equal to Ka, Kb, Kc, and, as $2\Delta = ax+by+cz$ $=\mathrm{K}(a^2+b^2+c^2)$, we have $\mathrm{K} = \dfrac{2\Delta}{(a^2+b^2+c^2)}$; in the figure let O be this point, and let DE be drawn so as to be bisected at O, then DE will be one of the diagonals in question; draw OF, OH parallel to a and c, then BF = OH $= \dfrac{x}{\sin B} = \dfrac{\mathrm{K}a}{\sin B}$, similarly BH $= \dfrac{\mathrm{K}c}{\sin B}$,



$\therefore \mathrm{DE}^2 = 4\mathrm{FH}^2 = 4(\mathrm{BF}^2+\mathrm{BH}^2-2\mathrm{BF}.\mathrm{BH}\cos B)$

$$= \frac{4\mathrm{K}^2}{\sin^2 B}(a^2+c^2-2ac\cos B) = \frac{4\mathrm{K}^2 b^2}{\sin^2 B}, \text{ therefore } \mathrm{DE} = \frac{2\mathrm{K}b}{\sin B}$$

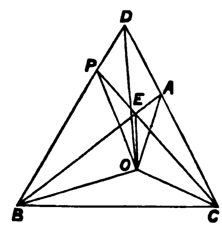
$$\left[\ldots\right] = \frac{4\Delta b}{(a^2+b^2+c^2)\sin B} = \frac{2abc}{(a^2+b^2+c^2)}.$$

The symmetry of the result shows that the three lines thus drawn are equal, each pair being the diagonals of one of the three rectangles referred to in the question.

7758. (By J Brill, B.A.)—If ABC be any triangle, and O a point within it; prove that

$$\frac{OA \cdot BC}{\sin(BOC-BAC)} = \frac{OB \cdot AC}{\sin(COA-CBA)} = \frac{OC \cdot AB}{\sin(AOB-ACB)}.$$

Solution by R. Lachlan, B.A.; R. Knowles, B.A.; *and others.*



Let the circles AOC, AOB cut AB, AC in E and D; and let CE, BD meet in P; then

$\angle PBC = AOB-ACB$, $\angle PCB = AOC-ABC$,

therefore $\angle BPC = BOC-BAC$.

Again, $\angle PBO = OAC$,

also $\angle ODB = OAB = OCP$;

therefore P, O, C, D are concyclic, therefore

$\angle BPO = OCA$;

hence the triangles PBO, CAO are similar,

therefore $\dfrac{OB}{BP} = \dfrac{OA}{AC}$, therefore $\dfrac{OA \cdot BC}{BC} = \dfrac{OB \cdot AC}{PB} = \dfrac{OC \cdot AB}{PC}$,

by symmetry, whence follows the result in the question.

[If $OA = x$, $OB = y$, $OC = z$, $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$,

we have $a^2 = y^2+z^2-2yz\cos\alpha$, $b^2 = z^2+x^2-2zx\cos\beta$............(1, 2),

$c^2 = x^2+y^2-2xy\cos\gamma$ ....(3),

$bc\sin A = ca\sin B = ab\sin C = yz\sin\alpha+zx\sin\beta+xy\sin\gamma$......(4),

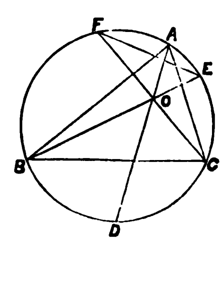
(2) + (3) − (1) gives $bc\cos A = x^2-xy\cos\gamma-zx\cos\beta+yz\cos\alpha$ ....(5).

Eliminating yz from (4), (5), we have, since $\alpha+\beta+\gamma = 2\pi$,

$$bc\sin(\alpha-A) = x^2\sin\alpha - xy\sin(\alpha+\gamma) - xz\sin(\alpha+\beta)$$
$$= x(x\sin\alpha+y\sin\beta+z\sin\gamma),$$

$$\therefore \frac{ax}{\sin(\alpha-A)} = \frac{abc}{x\sin\alpha+y\sin\beta+z\sin\gamma} = \frac{by}{\sin(\beta-B)} = \frac{cz}{\sin(\gamma-C)},$$

by symmetry.



Geometrically, if AO, BO, CO cut the circum-circle in D, E, F; then, from the triangles OEF, OBC, we have

$$\frac{EF}{BC} = \frac{EO}{OC} = \frac{EO \cdot OB}{B \cdot OC};$$

hence $\dfrac{EF}{AO \cdot BC} = \dfrac{FD}{BO \cdot CA} = \dfrac{DE}{CO \cdot AB}$;

also $\angle ECF = BOC-CEB$, when follows the result in the question.

This theorem enables us to find the distances

from the angular points of the triangle of the in-centre, es-centres, ortho-centre, and in fact of any point when the angles are known, and the angles when the distances are known.]

7802. (By W. J. GREENSTREET, B.A.)—Prove that the sum to infinity of the series $\log\frac{2 \cdot 4}{3^2}+\log\frac{4 \cdot 6}{5^2}+\log\frac{6 \cdot 8}{7^2}+\ldots$ is $\log\frac{\pi}{4}$.

Solution by W. T. MITCHELL, M.A.; G. G. STORR, B.A.; *and others.*

The sum $=\log\frac{2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \ldots}{3^2 \cdot 5^2 \cdot 7^2 \ldots}=\log \cdot \frac{2 \cdot 4 \cdot 4 \cdot 6 \ldots}{3 \cdot 3 \cdot 5 \cdot 5 \ldots}=\log\frac{\pi}{4}$, by WALLIS's theorem.

7795. (By C. E. McVICKER, B.A.)—Prove that the distance between the instantaneous centre of rotation of a movable line, and the centre of curvature of its envelope is, in any position, $dx/d\omega$, where x is the distance of any carried point on the line from the point of contact, and ω the angle of rotation.

Solution by W. J. C. SHARP, M.A.; *the* PROPOSER; *and others.*

The instantaneous centre for any position of the line is its point of contact with the envelope; if P and P′ be successive points upon this, $dx = \mathrm{PP'} = ds$ in the envelope, and, the successive positions of the line being the tangents at P and P′, $\rho = ds/d\omega = dx/d\omega$.

7794. (By J. BRILL, B.A.)—Prove that in any triangle

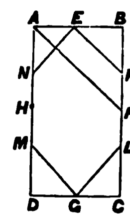
$$a^3\cos(\mathrm{B}-\mathrm{C})+b^3\cos(\mathrm{C}-\mathrm{A})+c^3\cos(\mathrm{A}-\mathrm{B}) = 3abc.$$

Solution by REV. D. THOMAS, M.A.; G. G. STORR, B.A.; *and others.*

$$\begin{aligned} a^3\cos(\mathrm{B}-\mathrm{C})+\ldots &= a\left[a\cos\mathrm{B}\, a\cos\mathrm{C}+a\sin\mathrm{B}\, a\sin\mathrm{C}\right]+\ldots \\ &= a\left[(b-c\cos\mathrm{A})(c-b\cos\mathrm{A})+bc\sin^2\mathrm{A}\right]+\ldots \\ &= a\left[bc-(b^2+c^2)\cos\mathrm{A}+bc\right]+\ldots = 6abc-bc\,(b\cos\mathrm{C}+c\cos\mathrm{B})-\ldots \\ &= 6abc-3abc = 3abc. \end{aligned}$$

7801. (By B. HANUMANTA RAU, M.A.)—Inscribe a regular hexagon in a rectangle whose sides are a and b; and find the ratio of a to b in order that the polygon may be also equiangular. [Suggested by Question 7636.]

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.



Let ABCD be the given rectangle, in which $AB = a$, $BC = b$, and, suppose $b > a$; let E, F, G, H be the middle points of the sides; on EF as base construct a triangle, EKF, whose vertex shall be on BC, and such that $EK = 2KF$; take $NH = HM = LF = FK$, and EKLGMN will be an *equilateral* hexagon; it will moreover be a *regular* hexagon if $\angle EKB = \frac{1}{3}\pi$, or if

$$KB = \tfrac{1}{2}KE = KF,$$

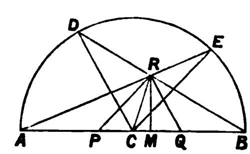
and therefore AF is parallel to EK, therefore

$$\frac{b}{2a} = \frac{FB}{BA} = \tan\tfrac{1}{6}\pi = \frac{1}{3^{\frac{1}{2}}}, \text{ or } 2a = 3^{\frac{1}{2}}b.$$

NOTE ON THE SOLUTIONS OF QUESTION 5672.

(See pp. 45—47 of this Volume.)

Mr. BIDDLE's result, $1/k^2l^2 \operatorname{cosec}\theta$, may be thus translated into terms of coordinates of R in respect of centre of circle:—In the figure, we have $AE = k$, $BD = l$, $\angle DCE = PRQ = \theta$, $CM = x$, $RM = y$, $AC = 1$; but



$$AE = \cos A \,.\, AB, \quad BD = \cos B \,.\, AB,$$

therefore $k = 2\cos A$, $l = 2\cos B$.

Also $\quad \sin A = y \,/\, [(1+x)^2 + y^2]^{\frac{1}{2}}$,

$$\sin B = y \,/\, [(1-x)^2 + y^2]^{\frac{1}{2}}, \quad \cos A = (1+x) \,/\, [(1+x)^2 + y^2]^{\frac{1}{2}},$$

$$\cos B = (1-x) \,/\, [(1-x)^2 + y^2]^{\frac{1}{2}}.$$

Moreover, $\operatorname{cosec}\theta = 1 / \sin\theta = 1 / \sin(180^\circ - \theta) = 1 / \sin(RPQ + RQP)$

$$= 1 / \sin 2(A+B) = 1 / 2\sin(A+B)\cos(A+B)$$

$$= 1 / [2\sin A\cos A(\cos^2 B - \sin^2 B) + 2\sin B\cos B(\cos^2 A - \sin^2 A)]$$

$$= \frac{(1-x^2)^2 + y^4 + 2(1+x^2)y^2}{4y(1-x^2-y^2)};$$

moreover, $k^2l^2 \operatorname{cosec}\theta = 16\cos^2 A\cos^2 B\operatorname{cosec}\theta = \dfrac{16(1-x^2)^2}{4(1-x^2-y^2)}$,

and $$\frac{1}{k^2l^2\operatorname{cosec}\theta} = \frac{y(1-x^2-y^2)}{4(1-x^2)^2},$$

which, to the constant 4 *près*, agrees with Colonel CLARKE's result.

7800. (By E. BUCK, B.A.)—Without involving the Integral Calculus, prove the formula $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \&c.$

Solution by G. G. STORR, B.A.; B. HANUMANTA RAU, M.A.; *and others.*

Since $\sin^{-1} 0 = 0$ and $\sin^{-1}(-x) = -\sin^{-1} x$, therefore $\sin^{-1} x$ is an odd function of x; hence, putting $\sin^{-1} x = Ax + B\frac{x^3}{3} + C\frac{x^5}{5} + \&c.$, and, differentiating, $\frac{1}{(1-x^2)^{\frac{1}{2}}} = A + Bx^2 + Cx^4 + \&c. = 1 + \frac{1}{2} \cdot x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \&c.$ (by Algebra). Hence, equating coefficients and substituting, we have the stated formula.

3826. (By J. B. SANDERS.)—The heights of the ridge and eaves of a house are 40 feet and 32 feet respectively, and the roof is inclined at 30° to the horizon. Find where a sphere rolling down the roof from the ridge will strike the ground, and also the time of descent from the eaves.

Solution by Professor MARTIN, M.A.; Professor WICKERSHAM; *and others.*

Let u be the velocity of the sphere at the eaves equal to the velocity of the sphere acquired by falling freely through 8 feet; and equal

$$(16 \times 32{\cdot}15945)^{\frac{1}{2}} = 22{\cdot}6837 \text{ feet.}$$

The velocity in a vertical direction at the eaves is $= 11{\cdot}3418$ feet. And the horizontal velocity at the eaves is $= u \cos 30° = 22{\cdot}6837 \times \frac{1}{2}\sqrt{3}$. Let t be the time of the sphere falling from the eaves. Then in the time t the sphere falls through $\frac{1}{2}ut$ by the constant velocity $\frac{1}{2}u$; and in the time t it falls through $\frac{1}{2}gt^2$ by the accelerated velocity commencing at the eaves, where $g = 32{\cdot}15945$ feet; then we have the equation

$$\tfrac{1}{2}gt^2 + \tfrac{1}{2}ut = 32 \text{ or } t^2 + \frac{ut}{g} = \frac{64}{g}, \text{ and } t = -\frac{u}{2g} + \left(\frac{u^2}{4g^2} + \frac{64}{g}\right)^{\frac{1}{2}};$$

and $t = -0{\cdot}352675 + 1{\cdot}454119 = 1{\cdot}101444$ seconds, the time of descent from eaves. And $tu \cos 30° = 1{\cdot}101441 \times 22{\cdot}6837 \times 0{\cdot}866025 = 21{\cdot}6375$ feet. And the sphere will strike the ground 21·6375 feet plus half the radius of the sphere from a vertical line from the eaves.

After the sphere leaves the eaves, it describes a parabola whose radius of curvature is 18·475 feet at the eaves; therefore a sphere of 2 or 3 feet radius, or less, will not touch the eaves after a radius of the sphere becomes perpendicular to the roof at the eaves. Friction and the resistance of the air are not considered.

7774. (By Professor WOLSTENHOLME, M.A., Sc.D.)—The lengths of the edges OA, OB, OC of a tetrahedron OABC are respectively

9·257824, 8·586, and 8·166; those of the respectively opposite edges BC, CA, AB are 8·996, 9·587, and 9·997. Prove that the dihedral angles opposite to OA and BC are equal to each other (each $= 7^\circ 19' 18''$). Denoting the lengths by a, b, c, x, y, z, and the dihedral angles respectively opposite by A, B, C, X, Y, Z, find what relation must subsist between a, b, c, x, y, z in order that A may be equal to X.

Solution by W. H. BLYTHE, M.A.

1. We have $\angle \mathrm{BAC} = 54^\circ 38' 32''$, $\mathrm{CAO} = 51^\circ 19' 17''{\cdot}6$, $\mathrm{BAO} = 52^\circ 47' 37''{\cdot}4$; and if we denote the dihedral angle opposite BC by M, and the above angles by p, q, r, and half their sum by s, then

$$4 \tan \tfrac{1}{2} \mathrm{M} = 10 + \tfrac{1}{2} [\mathrm{L} \sin (s-q) + \mathrm{L} \sin (s-r)] - \tfrac{1}{2} [\mathrm{L} \sin s + \mathrm{L} \sin (s-p)];$$

hence we obtain the stated result for M.

By a similar process, we find the dihedral opposite OA, from the values $\mathrm{OBA} = 59^\circ 10' 49''{\cdot}8$, $\mathrm{OBC} = 55^\circ 17' 24''{\cdot}3$, $\mathrm{CBA} = 60^\circ 21' 29''{\cdot}4$.

2. $$\sin \mathrm{A} = \frac{(3 \text{ volume})\, x}{2\,(\text{area ABC})\,(\text{area OBC})}, \quad \sin \mathrm{X} = \frac{(3 \text{ volume})\, a}{2\,(\text{area OBA})\cdot(\text{area CAO})},$$

and, if these angles are equal,

$$\begin{aligned} &a^2 (b+c+x)(b+c-x)(b+x-c)(x+c-b) \\ &\qquad \times (x+y+z)(x+y-z)(z+x-y)(y+z-x) \\ &= x^2 (a+b+z)(a+b-z)(a+z-b)(b+z-a) \\ &\qquad \times (a+c+y)(a+c-y)(a+y-c)(y+c-a). \end{aligned}$$

7652. (By G. HEPPEL, M.A.)—Show that the square root of

$$\begin{aligned} 2\mathrm{E} \equiv 2\,(&1 + \cos\alpha \cos\beta - \cos\alpha \cos\gamma - \cos\alpha \cos\delta - \cos\beta \cos\gamma - \cos\beta \cos\delta \\ &+ \cos\gamma \cos\delta - \sin\alpha \sin\beta \sin\gamma \sin\delta + \cos\alpha \cos\beta \cos\gamma \cos\delta) \end{aligned}$$

is $$2 \cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma-\delta) \sim 2 \cos \tfrac{1}{2}(\alpha-\beta) \cos \tfrac{1}{2}(\gamma+\delta).$$

Solution by B. HANUMANTA RAU, M.A.; SARAH MARKS; *and others.*

$$\begin{aligned} 2\mathrm{E} &= (1+\cos\alpha)(1+\cos\beta)(1-\cos\gamma)(1-\cos\delta) \\ &\qquad + (1-\cos\alpha)(1-\cos\beta)(1+\cos\gamma)(1+\cos\delta) \\ &\quad - 2 \sin\alpha \sin\beta \sin\gamma \sin\delta \\ &= (4 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}\delta)^2 + (4 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\delta)^2 \\ &\quad - 2 \cdot 4 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}\delta \times 4 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\delta, \end{aligned}$$

$$\therefore (2\mathrm{E})^{\frac{1}{2}} = 4 (\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \sin \tfrac{1}{2}\delta \sim \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\delta) = \&c.$$

4165. (By J. CONWILL.)—A fish is floating in a cubical glass tank filled with water, with its head in one corner, and its tail towards the one diagonally opposite; describe the appearance which will be presented to an eye looking towards the corner in the direction of the length of the fish, and in the same horizontal plane with it.

Solution by the Rev. J. L. Kitchin, M.A.; A. Martin, M.A.; *and others.*

The fish, looked at along the diagonal in which it is placed, will be seen by light refracted out of a denser medium into a rarer. Hence each point in the two sides will appear shifted from its position *from* the diagonal, and the nearer the point the greater will the displacement appear, because it is seen under a greater angle. The two sides will then appear to recede from the diagonal, and the appearance presented is that of two fish united by their heads at the further corner of the cube.

4737. (By Professor Artemas Martin, M.A., Ph.D.)—Three equal circles, each 4 inches in diameter, are drawn at random on a circular slate whose diameter is 12 inches; find the probability that each circle intersects the other two.

Solution by D. Biddle.

In order that three complete circles, each 4 inches in diameter, may lie on a circular slate 12 inches in diameter, the centre of each must be within the circumference of a circle concentric with the slate and 8 inches in diameter. Moreover, in order that two of the smaller circles may intersect each other, their centres must lie within a radius of 4 inches of each other; and the probability of this occurring is the proportion borne to the entire area of the 8-inch circle by the mean space cut off by another 8-inch circle whose centre may be anywhere between the former's centre and circumference. This mean space is the average for all positions within the 8-inch circle, and is found by taking a series of segments from half-radius to radius in height, doubling each, and multiplying also by the difference between the height and radius, which difference is half the distance of the centre of the invading circle from the centre of the slate, and is proportioned to the ring (concentric with the slate) for all parts of which the particular segment will serve; and we finally divide the sum of the several products by the sum of the multipliers above referred to. This gives $\frac{2272}{3927} = \cdot 57856$, as the probability that a 4-inch circle drawn at random on the slate will cut one of similar size also drawn at random on the slate. Now, if A, B, C be the three 4-inch circles, the above is the probability that A will cut B, that B will cut C, that C will cut A. But, as these events must concur to produce the mutual intersection of all three circles, therefore $(\cdot 57856)^3$ is the probability required $= \cdot 19366$, or rather more than $\frac{6}{31}$.

6746. (By Âsûtosh Mukhopâdhyây.)—A certain number of candidates apply for a situation, to whom the voters attribute every degree of merit between the limits 0 and ψ; find the mean value of all the candidates' merits.

Solution by H. Fortey, M.A.; G. Eastwood, M.A.; *and others.*

Let n be the number of candidates, and $t_1, t_2, t_3 \ldots t_r$ their respective

merits, t_1 being the merit attributed to the candidate whom the voters deem most worthy, t_2 the merit ascribed to the next candidate, and t_r the merit attributed to the candidate in the rank r.

First, suppose that t_1 may have any positive value comprised between the limits 0 and ψ, and that the mean of all such positive values of t_1 is required. Let w be the whole number of the values, and Σt_1 their sum; then, if M be the required mean, $M = \frac{\Sigma t_1}{w}$. Suppose that the values of t_1 receive equal increments Δ, and that they become 0, Δ, 2Δ, $3\Delta \ldots \psi$; then $w = \frac{\Sigma\Delta}{\Delta}$; in this case $M = \frac{\Delta\Sigma t_1}{\Delta\Sigma}$; but, because t_1 is supposed to receive all values between 0 and ψ, the quantity Δ may be considered indefinitely small, and may consequently be represented by dt_1. Hence

$$M = \frac{\int_0^\psi t_1 dt_1}{\int_0^\psi dt_1} = \frac{\frac{1}{2}\psi^2}{\psi} = \frac{\psi}{2};$$

hence the mean of all these values is the mean of the two extreme values.

Next suppose t_1, t_2 to be the merits of two candidates according to the opinion of any voter, t_1 being the merit attributed to the more worthy candidate, t_1 therefore being greater than t_2. Here t_2 may have any values between 0 and t_1; and t_1 any values between $t_2 = 0$ and $t_2 = t_1$. The aggregate of all the values of t_1 contained between $t_2 = 0$ and $t_2 = t_1$ is equal to $\frac{t_1 \int_0^{t_1} dt_2}{dt_2}$, and the sum of all the values of t_1 is equal to $\frac{\int_0^\psi \left(t_1 dt_1 \int_0^{t_1} dt_2\right)}{dt_1 dt_2}$; the number of these values is $\frac{\int_0^\psi dt_1 \int_0^{t_1} dt_2}{dt_1 dt_2}$; hence the mean value of t_1 is $\frac{\int_0^\psi \int_0^{t_1} t_1 dt_1 dt_2}{\int_0^\psi \int_0^{t_1} dt_1 dt_2}$.

If t_r be the merit attributed to the candidate in rank r,

$$\text{the mean value of } t_1 = \frac{\int_0^\psi \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} t_r dt_1 dt_2 \ldots dt_n}{\int_0^\psi \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} dt_1 dt_2 \ldots dt_n}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n-r} \cdot \frac{1}{1-r+2} \cdots \frac{1}{n+1} \psi^{n+1}}{\frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n} \psi^n} = \frac{n-r+1}{n+1} \psi.$$

Now, if n be an odd number, $t_{\frac{1}{2}(n+1)}$ will be the middle term, and its mean value $\frac{1}{2}\psi$. If n be an even number, $t_{\frac{1}{2}n}$, $t_{\frac{1}{2}(n+2)}$ are the two middle terms; their values are $\dfrac{\psi}{2}+\dfrac{\frac{1}{2}\psi}{n+1}$ and $\dfrac{\psi}{2}-\dfrac{\frac{1}{2}\psi}{n+1}$, and the mean of these is $\frac{1}{2}\psi$. The sum of a mean value is $\frac{1}{2}\psi n$, and, therefore, the mean of all the values is $\frac{1}{2}\psi$.

7780. (By the Rev. T. R. Terry, M.A.)—Prove that the mean value of the fourth powers of the distances from the centre of all points inside an ellipsoid whose axes are $2a$, $2b$, $2c$, is

$$\mathrm{A} = \tfrac{1}{35}\left[(a^2+b^2+c^2)^2+2\,(a^4+b^4+c^4)\right].$$

Solution by Nilkanta Sarkar, M.A.; D. Edwardes; *and others.*

We have $$\mathrm{A} = \tfrac{1}{8}\pi abc\,\mathrm{A} = \iiint (x^2+y^2+z^2)^2\,dx\,dy\,dz,$$

over the positive compartment. Integrating with respect to z, and then putting, successively, $x = ax' = ar\cos\theta$, $y = by' = br\sin\theta$, the dexter becomes

$$abc\int_0^1\int_0^{(1-x^{2})^{\frac{1}{2}}}\left[(1-x'^2-y'^2)^{\frac{1}{2}}(a^2x'^2+b^2y'^2)^2+\tfrac{2}{3}(a^2x'^2+b^2y'^2)\,c^2\,(1-x'^2-y'^2)^{\frac{3}{2}}\right.$$
$$\left.+\tfrac{1}{5}c^4\,(1-x'^2-y'^2)^{\frac{5}{2}}\right]dx'\,dy' =$$

$$abc\int_0^{\frac{1}{2}\pi}\int_0^1\left[r^4\,(1-r^2)^{\frac{1}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)^2+\tfrac{2}{3}c^2r^2\,(1-r^2)^{\frac{3}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)\right.$$
$$\left.+\tfrac{1}{5}c^4\,(1-r^2)^{\frac{5}{2}}\right]r\,dr\,d\theta.$$

Putting $r = \sin\phi$, and integrating with respect to ϕ, this becomes

$$abc\int_0^{\frac{1}{2}\pi}\left[\frac{8}{3\,.\,35}(a^2\cos^2\theta+b^2\sin^2\theta)^2+\frac{4c^2}{3\,.\,35}(a^2\cos^2\theta+b^2\sin^2\theta)+\frac{1}{35}c^4\right]d\theta,$$

and, integrating with respect to θ, we have

$$\tfrac{1}{8}\pi abc = \tfrac{1}{8}\pi\,.\,\tfrac{1}{35}abc\left[(a^2+b^2+c^2)^2+2\,(a^4+b^4+c^4)\right].$$

[By Lejeune Dirichlet's Theorem, as given in Todhunter's chapter on Definite Integrals, we have, by taking integrations over one-eighth of the ellipsoid,

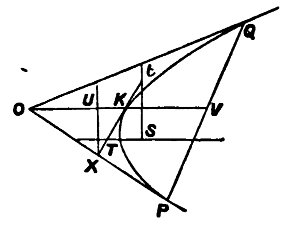
$$\iiint x^4\,dx\,dy\,dz = \frac{a^5bc}{8}\,\frac{\Gamma(\frac{5}{2})\,.\,\Gamma(\frac{1}{2})\,.\,\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})} = \frac{\pi a^5bc}{70},$$

$$\iiint y^2z^2\,dx\,dy\,dz = \frac{ab^3c^3}{8}\,\frac{\Gamma(\frac{1}{2})\,.\,\Gamma(\frac{3}{2})\,.\,\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{2})} = \frac{\pi ab^3c^3}{210};$$

therefore $\tfrac{1}{8}\pi abc\,\mathrm{A} = \tfrac{1}{210}\pi abc\left[3\,(a^4+b^4+c^4)+2\,(b^2c^2+c^2a^2+a^2b^2)\right]$; whence the result follows.]

7792. (By Asparagus.)—The tangent at any point of a parabola meets the axis in T and the latus rectum in t; prove that Tt is equal to one-fourth of the parallel normal chord.

Solution by the Rev. D. THOMAS, M.A.; G. G. STORR, B.A.; *and others.*



1. The length of a normal chord inclined at an angle θ to the axis of the parabola $y^2 = 4ax$ is $4a \operatorname{cosec}^2\theta \sec\theta$, and it is easily shown that $Tt = a \operatorname{cosec}^2\theta \sec\theta$.

2. *Otherwise.*—Let K be the point of contact of Tt, KU a perpendicular on the directrix, PQ the parallel chord normal at P, O the pole of PQ, X the pole of PK; then, two tangents meeting at right angles in X, X is a point on the directrix, and the projections of tT, KX on the axis are ST, KU, and are equal; or $Tt = XK = \frac{1}{4}PV = \frac{1}{4}PQ$.

7532. (By Â. MUKHOPÂDHYÂY.)—Prove that

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \sqrt{\left\{1 - \frac{1}{\sqrt{2}\,.\,\sin\theta}\right\}}\, d\theta = \pi\sqrt{[1 - 2^{-\frac{1}{2}}]}\left[1 - \frac{2}{\pi}\,\mathrm{F}\,(\sqrt{2} - 1)\right].$$

[See CAYLEY's *Elliptic Functions*, pp. 120—4.]

Solution by Professor ARTEMAS MARTIN, M.A., Ph.D.

If $\sin\theta = \dfrac{1-x^2}{1+x^2}$, $\cos\theta = \dfrac{2x}{1+x^2}$, $\cos\theta\, d\theta = -\dfrac{4x\,dx}{(1+x^2)^2}$; $d\theta = -\dfrac{2dx}{1+x^2}$,

and $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \sqrt{\left\{1 - \frac{1}{\sqrt{2}\,.\,\sin\theta}\right\}}\, d\theta = \sqrt{(4 - 2\sqrt{2})}\int_0^{\sqrt{2}-1} \frac{\sqrt{(3 - 2\sqrt{2} - x^2)}\,dx}{(1+x^2)\sqrt{(1-x^2)}}$

(putting $\sqrt{2} - 1 = e$, $ey = x$),

$$= \sqrt{(4-2\sqrt{2})}\int_0^1 \frac{e^2\sqrt{(1-y^2)}\,dy}{(1+e^2y^2)\sqrt{(1-e^2y^2)}}$$

$$= \sqrt{(4-2\sqrt{2})}\int_0^1 \frac{e^2(1-y^2)\,dy}{(1+e^2y^2)\sqrt{(1-y^2)}\sqrt{(1-e^2y^2)}}$$

$$= \sqrt{(4-2\sqrt{2})}\int_0^1 \left[\frac{(1+e^2)\,dy}{(1+e^2y^2)\sqrt{(1-y^2)}\sqrt{(1-e^2y^2)}} - \frac{dy}{\sqrt{(1-y^2)}\sqrt{(1-e^2y^2)}}\right]$$

$$= \sqrt{(4-2\sqrt{2})}\left[(1+e^2)\,\Pi\,(e^2, e, y) - \mathrm{F}\,(e, y)\right]_0^1$$

$$= \sqrt{(4-2\sqrt{2})}\,[(4-2\sqrt{2})\,\Pi\,(3-2\sqrt{2},\ \sqrt{2}-1) - \mathrm{F}\,(\sqrt{2}-1)].$$

The equation in CAYLEY's *Elliptic Functions* referred to is, writing e for k, and putting $\qquad a = (1+n)\left(1 + \dfrac{e^2}{n}\right),$

$$\Pi\,(n, e, \phi) + \Pi\left(\frac{e^2}{n}, e, \phi\right) - \mathrm{F}\,(e, \phi) = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{\sqrt{a}\tan\phi}{\sqrt{(1-e^2\sin^2\phi)}}\right).$$

When $n = e^2$ and $\phi = \frac{1}{2}\pi$, we have

$$\Pi(e^2, e) + \Pi(1, e) - F(e) = \frac{\pi}{2 \ldots [2(1+e^2)]},$$

which by mistake I wrote $2\Pi(e^2, e) - F(e) = \dfrac{\pi}{2(1+e^2)}$.

This relation, however, is of no use here, as it will not enable us to get rid of the third-order functions. We could, by it, eliminate $F(e)$, and thus obtain a result containing only one kind of elliptic functions.

[Professor MARTIN states that he suspects this question was suggested to the PROPOSER by his own solution of Problem 200 in the *Mathematical Visitor*, Vol. ii., No. 1, p. 20, the result of which is erroneous, owing to a mistake made in attempting to eliminate $\Pi(e^2, e)$ by the formula referred to in CAYLEY's *Elliptic Functions*.]

1208 & 6695. (By the EDITOR.)—(1208.) Show that the values of x, y, z, from the equations

$$x^2 + 4xy + 6y^2 = 28,\quad x^2 + 4xz + 14z^2 = 60,\quad 3y^2 + 2yz + 7z^2 = 40 \ldots (1, 2, 3),$$

are given by
$$x^2 = (\pm\sqrt{5}-1)(\pm 5\sqrt{2}-6)(\pm\sqrt{10}-2),$$
$$y^2 = \tfrac{1}{3}(\pm\sqrt{5}+1)(\pm 5\sqrt{2}-6)(\pm\sqrt{10}+2),$$
$$z^2 = \tfrac{1}{14}(\pm\sqrt{5}+1)(\pm 5\sqrt{2}+6)(\pm\sqrt{10}-2).$$

(6695.) The sides of a triangle are 40, 30, 14, and x^2, y^2, z^2 are the radii of three circles respectively inscribed in the angles opposite to the sides 40, 30, 14, such that each touches the other two and two sides of the triangle; show that the values of these radii are given by

$x^2 = 1{\cdot}539,\ y^2 = 2{\cdot}982,\ z^2 = 3{\cdot}583,\ x^2 = 83{\cdot}416,\ y^2 = 8{\cdot}194,\ z^2 = 1{\cdot}278,$
$x^2 = 49{\cdot}163,\ y^2 = 13{\cdot}901,\ z^2 = {\cdot}110,\ x^2 = 17{\cdot}893,\ y^2 = {\cdot}257,\ z^2 = 5{\cdot}958,$

of which triads of values the first gives the radii of the *inscribed* circles, and the other three those of the three triads of *escribed* circles.

Solutions by (1) *the* PROPOSER; (2) *the late* S. BILLS.

1. The equations in Question 1208 may be written thus:—

$$x^2\cot\alpha + y^2\cot\beta + 2xy = 4(\cot\alpha + \cot\beta),$$
$$x^2\cot\alpha + z^2\cot\gamma + 2xz = 4(\cot\alpha + \cot\gamma),$$
$$y^2\cot\beta + z^2\cot\gamma + 2yz = 4(\cot\beta + \cot\gamma),$$

α, β, γ being three angles such that $\frac{1}{2}\tan\alpha = 3\tan\beta = 7\tan\gamma = 1 \ldots$(A).

The solution of the above equations is (see HYMERS's *Trigonometry*, p. 153, 3rd edition) $\lambda^2x^2 = \mu^2y^2 = \nu^2z^2 = 2\lambda\mu\nu$ ....(B),

where $\lambda - \tan\frac{1}{2}\alpha = \mu - \tan\frac{1}{2}\beta = \nu - \tan\frac{1}{2}\gamma = 1$ ....(C).

From the groups of equations (A), (C), we have

$$\lambda = \tfrac{1}{2}(\pm\sqrt{5}+1),\quad \mu = \pm\sqrt{10}-2,\quad \nu = \pm 5\sqrt{2}-6;$$

and then (B) gives the results stated in the question.

Multiplying out the radicals, the values of one triad are

$$x^2 = 38-20\sqrt{2}+2\sqrt{5}-4\sqrt{10},\quad y^2 = \tfrac{1}{6}(38-20\sqrt{2}-2\sqrt{5}+4\sqrt{10}),$$
$$z^2 = \tfrac{1}{14}(38+20\sqrt{2}-2\sqrt{5}-4\sqrt{10}).$$

A verification may be obtained as follows :—

$$xy = 2\sqrt{50}-12,\quad xz = 2\sqrt{10}-4,\quad yz = \sqrt{5}+1;$$
$$x^2+4xy+6y^2 = (\sqrt{50}-6)(\,2\sqrt{50}+12) = 28,$$
$$x^2+4xz+14z^2 = (\sqrt{10}-2)(10\sqrt{10}+20) = 60,$$
$$3y^2+2yz+7z^2 = (\sqrt{5}\ +1)(10\sqrt{5}\ -10) = 40.$$

2. *Otherwise* :—From (1) + (2) − 2 (3) we have

$$x^2+2xy+2xz-2yz = 4 \quad\text{..........................}(4).$$

Let $y = mx$, $z = nx$; then equations (1), (2), (4) become

$$x^2(1+4m+6m^2) = 28,\quad x^2(1+4n+14n^2) = 60 \quad\text{...........}(5,\ 6),$$
$$x^2(1+2m+2n-2mn) = 4 \quad\text{..........................}(7).$$

From (5) ÷ (6) and (5) ÷ (7) we have

$$\frac{1+4m+6m^2}{1+4n+14n^2} = \tfrac{7}{15},\quad \frac{1+4m+6m^2}{1+2m+2n-2mn} = 7 \quad\text{...........}(8,\ 9).$$

From (9), $n = \dfrac{3+5m-3m^2}{7(m-1)}$; and, this being substituted in (8), the result is

$$36m^4-24m^3-34m^2-4m+1 = 0 \quad\text{....................}(10).$$

Assume
$$(10) = (6m^2+\mathrm{A}m+1)(6m^2+\mathrm{B}m+1)$$
$$= 36m^4+6(\mathrm{A}+\mathrm{B})\,m^3+(\mathrm{AB}+12)\,m^2+(\mathrm{A}+\mathrm{B})\,m+1 \quad\text{........}(11);$$

then, comparing the coefficients of (10) and (11), we have $\mathrm{A}+\mathrm{B} = -4$, $\mathrm{AB} = -46$; whence $\mathrm{A} = -2+5\sqrt{2}$ and $\mathrm{B} = -2-5\sqrt{2}$. Hence the four roots of (10) are those of the two quadratics $6m^2-(2\pm5\sqrt{2})\,m+1 = 0$; these roots give, therefore, for $12m$, the values

$$2+5\sqrt{2}+2\sqrt{5}+\sqrt{10},\quad 2-5\sqrt{2}+2\sqrt{5}-\sqrt{10},\quad 2-5\sqrt{2}-2\sqrt{5}+\sqrt{10},$$
$$2+5\sqrt{2}-2\sqrt{5}-\sqrt{10}.$$

Hence putting, for shortness,

$$a = 19+10\sqrt{2}+\sqrt{5}+2\sqrt{10},\quad b = 19+10\sqrt{2}-\sqrt{5}-2\sqrt{10},$$
$$c = 19-10\sqrt{2}+\sqrt{5}-2\sqrt{10},\quad d = 19-10\sqrt{2}-\sqrt{5}+2\sqrt{10},$$

the values of x^2, y^2, z^2 are

$$x^2 = 2c,\quad y^2 = \tfrac{1}{3}d,\quad z^2 = \tfrac{1}{7}b \quad\text{......................}(12),$$
$$x^2 = 2a,\quad y^2 = \tfrac{1}{3}b,\quad z^2 = \tfrac{1}{7}d \quad\text{......................}(13),$$
$$x^2 = 2b,\quad y^2 = \tfrac{1}{3}a,\quad z^2 = \tfrac{1}{7}c \quad\text{......................}(14),$$
$$x^2 = 2d,\quad y^2 = \tfrac{1}{3}c,\quad z^2 = \tfrac{1}{7}a \quad\text{......................}(15).$$

The given equations (in 1208) may be otherwise solved as follows :—

Put $84-x^2 = n^2$; then, from (1), (2), we have $6y = -2x\pm n\sqrt{2}$, $14z = -2x\pm n\ldots10$; and, these values of y, z being substituted in (3), the resulting reduced equation in x is

$$x^4-4(19\pm\sqrt{5})\,x^2+168(3\mp\sqrt{5}) = 0.$$

In a similar manner, we find for y and z the equations

$$y^4-\tfrac{2}{3}(19\pm2\ldots10)\,y^2+\tfrac{28}{9}(7\pm3\ldots10) = 0,$$
$$z^4-\tfrac{2}{7}(19\pm10\ldots2)\,z^2+\tfrac{12}{49}(43\pm30\ldots2) = 0.$$

The values of x^2, y^2, z^2 found from these are the same as those given above.

7585 & 7616. (By W. J. C. SHARP, M.A.)—(7585.) If two straight lines cut the sides of a triangle ABC in the points D and D′, E and E′, F and F′, respectively, and points d and d' be taken in BC harmonically conjugate to D and D′, e and e' in CA conjugate to E and E′, and f and f' in AB conjugate to F and F′, prove that each of the sets of six points (d, d', E, E′, F, F′), (D, D′, e, e', F, F′), (D, D′, E, E′, f, f'), and (d, d', e, e', f, f') will lie on a conic, and be such that the lines drawn to them from the opposite vertices form two pencils, each composed of three concurrent lines.

(7616.) If the vertices of the triangle of reference be joined to a point $(\alpha_1, \beta_1, \gamma_1)$, and a circle be described through the three points in which these lines intersect the opposite sides, prove that (1) the point of concurrence of the other three lines drawn from the angles to the intersections of the circle with the opposite sides is determined by the equations

$$\alpha_1\alpha\,(\beta_1\sin B+\gamma_1\sin C)(\beta\sin B+\gamma\sin C)=\beta_1\beta\,(\gamma_1\sin C+\alpha_1\sin A)$$
$$\times(\gamma\sin C+\alpha\sin A)=\gamma_1\gamma\,(\alpha_1\sin A+\beta_1\sin B)(\alpha\sin A+\beta\sin B);$$

and (2) if $\alpha_1=\alpha$, $\beta_1=\beta$, and $\gamma_1=\gamma$, these equations determine the points of concurrence of lines drawn from the vertices to the opposite points of contact of the inscribed and escribed circles.

Solution by the PROPOSER; R. KNOWLES, B.A.; *and others.*

Let $ax^2+by^2+cz^2-2fyz-2gzx-2hxy=0$ be the equation to the two straight lines; then the points (d, d'), (e, e'), (f, f') are determined by

$$\left.\begin{array}{l}(x=0,\ by^2+2fyz+cz^2=0)\\(y=0,\ cz^2+2gzx+ax^2=0)\\(z=0,\ ax^2+2hxy+by^2=0)\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

So that these all lie on the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0;$$

and, since $abc-2fgh-af^2-bg^2-ch^2=0$, this meets the sides of the triangle in points which are the feet of two sets of concurrent lines through the vertices. (See Question 6813, *Reprint*, Vol. 37, p. 21.)

If (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ be the points of concurrence, the equations (A) are equivalent to $\quad\left(\dfrac{y}{\beta}-\dfrac{z}{\gamma}\right)\left(\dfrac{y}{\beta_1}-\dfrac{z}{\gamma_1}\right)=0$, &c.,

and therefore

$$a:b:c:-2f:-2g:-2h=\frac{1}{\alpha\alpha_1}:\frac{1}{\beta\beta_1}:\frac{1}{\gamma\gamma_1}:\left(\frac{1}{\beta\gamma_1}+\frac{1}{\gamma_1\beta}\right):\text{\&c.};$$

and, if the conic be a circle,

$$b\sin^2C+c\sin^2B-2f\sin B\sin C=c\sin^2A+a\sin^2C-2g\sin C\sin A$$
$$=a\sin^2B+b\sin^2A-2h\sin A\sin B,$$

$$\therefore\left(\frac{\sin C}{\beta}+\frac{\sin B}{\gamma}\right)\left(\frac{\sin C}{\beta_1}+\frac{\sin B}{\gamma_1}\right)=\left(\frac{\sin A}{\gamma}+\frac{\sin C}{\alpha}\right)\left(\frac{\sin A}{\gamma_1}+\frac{\sin C}{\alpha_1}\right)=\text{\&c.},$$

which proves Question 7616 (1), if $\alpha_1=\alpha$, &c.; also

$$\frac{\sin C}{\beta}+\frac{\sin B}{\gamma}=\pm\left(\frac{\sin A}{\gamma}+\frac{\sin C}{\alpha}\right)=\pm\left(\frac{\sin B}{\alpha}+\frac{\sin A}{\beta}\right),$$

which give the four points required.

London: Printed by C. F. Hodgson & Son, Gough Square, Fleet Street, E.C.

APPENDIX.

RATIO RATIONIS:

Or that primary faculty of human nature which finds exercise alike in Logic, in Induction, and in the various processes of Mathematics.

AN ESSAY BY D. BIDDLE.

ALL things admit of both *comparison* and *contrast*. No two things are *equal* in all respects, for in that case they would be indistinguishable, and could not be known to be two; nor are any two things in all respects *unequal*, for in that case they could not exist together even in thought. The mind cannot conceive of any two things which have not some qualities in common, and also some qualities peculiar to each. 0 and ∞, the infinite and the infinitesimal, are very unlike each other, but they agree in concerning quantity, and in being at the extremes of it.

By the aid of perception and memory, the mind makes a natural classification (more or less correct) of all things coming within its range; and (taking the term in its widest signification) *Logic*, as a science, deals with the laws which govern this classification, and, as an art, teaches us to compare and contrast methodically, showing how, by successive steps, intricate comparisons and contrasts may be rendered more and more simple and intelligible, and how, on the other hand, upon a few simple facts we may build a superstructure, apparently far exceeding them in magnitude and importance, but consisting in reality only of deductions from them.

That which characterises a conscious being is the possession of a power to *feel*. This *sentient power* receives, from force transmitted to it, various *impressions*, which are agreeable or disagreeable; and it exerts an influence in promoting or preventing their recurrence accordingly. It would appear that a disagreeable impression is produced by anything which interferes with the harmonious exercise of the being's powers, and that all other impressions are agreeable.

The powers which are in immediate relation with the sentient power are the only ones of whose actions it is ever properly cognizant. Through them it receives all the force by which it is impressed, and upon them also it reacts. They communicate with (or form part of) a special organism, by which force is transformed into what are called *ideas*, and which, accordingly, we may term the *idea-framing organ*. Every impression which the sentient power receives is communicated to it by the idea-framing organ. Undisturbed by this organ, it is not conscious, we have reason to believe, even of its own existence. For self-consciousness is the result of a complex operation. When the sentient power has an idea presented to it, it instantaneously reacts upon the idea-framing organ, and gives rise to another idea. This contains a notion of the primary idea,

and also of its having produced an impression upon the sentient power. And therefore, when it is itself presented, it renders the sentient power indirectly self-conscious; the sentient power then *feels that it feels* (or that it *has felt*), and this is its self-consciousness. The sentient power perceives nothing respecting itself, but in this indirect manner. Moreover, the tendency of the evidence within our reach is to prove that the sentient power is dependent upon the idea-framing organ, not only for the *reception* of impressions, but for their *continuance* also; in other words, that it does not itself retain impressions, but, like a mirror, loses them instantaneously when that by which they are communicated ceases to operate. It is owing to this qualification of the sentient power (a qualification perfect of its kind, though apparently negative) that ideas can be received by it in such quick succession, without interfering with one another, or producing the effect of dissolving views.*

The Reason depends for its data upon Perception (internal and external) and Memory (including habit), and these are, from various causes, so different in different people that it is scarcely to be wondered at, that the testimony of two persons so seldom coincides regarding any occurrence, even when they have had, to all outward seeming, the same opportunity of forming an opinion. Wilful liars are by no means those who are least capable of telling the truth, although they are certainly those who when discovered are in future least credited.

One of the most potent factors in the production of false impressions is that *habit* (referred to above as included under Memory) which leads us, often prematurely and falsely, to fill in outlines of sensation (visual, aural, or otherwise) with what to us have been their usual details. It is this habit that the conjuror takes advantage of, and may be instanced by crossing the middle and forefinger and placing a marble between, when instantly a sensation is felt as of two marbles being there, the two fingers being touched on the sides usually farthest apart. It is very common to mistake a person who is seen in the distance. I have known a fellow-student dart from my side, and give a vigorous pat on the back to a perfect stranger, in the belief that he was thus warmly greeting an old friend. He was, of course, undeceived when the stranger turned round.

Then there are the more subtle illusions which defy the more ordinary means to dispel, and which have been known to hold even the civilised world enthralled. I refer to such as that which refers the rising and setting of the sun to his own movement, and such as that which makes plants and animals to grow of themselves. These things should teach us to exercise extreme caution in regard to even the most commonly received notions. The last half of most men's lives is engaged in unlearning what has been acquired during the first.

Now, in his excellent treatise on Logic, Whately says:—"Complaints have been made that Logic leaves untouched the greatest difficulties, and those which are the sources of the chief errors in reasoning, viz., the ambiguity or indistinctness of terms, and the doubts respecting the degrees of evidence in various propositions: an objection which is not to be removed by any such attempt as that of Watts to lay down 'rules for forming clear ideas, and for guiding the judgment'; but by replying that no art is to be censured for not teaching more than falls within its

---

* A considerable portion of the above is taken from "The Spirit Controversy" (Williams & Norgate), and was written by me as far back as 1867.

province, and indeed more than can be taught by any conceivable art. Such a system of universal knowledge as should instruct us in the full meaning of every term, and the truth or falsity, certainty or uncertainty, of every proposition, thus superseding all other studies, it is most unphilosophical to expect or even to imagine. And to find fault with Logic for not performing this, is as if one should object to the science of optics for not giving sight to the blind; or as if (like the man of whom Warburton tells a story in his 'Divine Legation') one should complain of a reading-glass for being of no service to a person who had never learned to read."

This is perfectly correct, but at the same time, when we leave symbols and attempt to reduce Logic to practice, we find that a syllogism is only a link in a chain of argument; and it is a fact never to be forgotten that a chain is no stronger than its weakest link. Every proposition, every premise, nay every well-defined perception, is the result of a whole series of syllogisms. For instance, I cannot truthfully say that "This which I see before me is a red ball," until my mind (by unconscious cerebration, it may be) has come to at least as many conclusions as there are words in the proposition.

Moreover, Logic makes use of universal affirmatives and negatives, whereas in practice we have to reduce these to strong probabilities at most; or, in case we make use of the syllogistic form, we have to insert the qualifying "if." Thus, Enoch and Elijah are men; therefore, *if* all men are mortal, Enoch and Elijah are mortal.

But let us consider the fundamental principle upon which the reason proceeds. The most intimate faculty of the human mind (next to that of bare *feeling*, above referred to) consists in the detection (however imperfect and undefined) of similarities and dissimilarities in the various objects of which it takes cognizance. This in its simplest form is a matter of impression or perception which defies further analysis. For, though we can perceive the difference between the impressions produced, for instance, by two colours such as red and blue, we cannot adequately describe the difference, much less the impressions themselves; and, for aught we know, our impression of red may be totally unlike that produced on another person, and this without any colour-blindness either on our part or his. It matters little, provided we can distinguish red from other colours, as well as our fellows. But it is more than probable that the impression produced by red is compound. If therefore it is difficult to describe the compound impression, how impossible must it be to describe the simpler impressions which compose it!

It is the same with all elementary impressions: we cannot describe them to other persons. But we can distinguish between them, and we can select and classify objects which produce various combinations of them: we can also on the same principle classify events.

Now the primary and fundamental axiom, as laid down by Euclid, is, that "things which are equal to the same thing are equal to one another." But this is apt to mislead if taken as indicating the simplest act of the reason. For the human mind can, in numberless instances, perceive the equality of two things without reference to any third thing, and could apparently recognize that equality, if no third thing of the kind existed. The human mind starts with a recognized (or at least assumed) equality between things, and the converse of Euclid's axiom would more accurately describe its mode of procedure, viz., that "things which are equal to one another are equal to the same thing." This can be variously rendered: "The equality of equals to other things is coextensive;" "That which can be

predicated of one of two things which are equal, can with equal truth be predicated of the other;" "Of two things which in any given respect are equal to each other, one cannot be equal and the other unequal, *in that respect*, to any third thing."

This, in reality, is the axiom underlying the construction and proof in the first proposition of Euclid. For when he says, "From the centre A, at the distance AB, describe the circle BCD,"—unless we regard as a species of *hocus pocus* the postulate demanding that such a thing may be considered possible,—we must imagine it as done by something comparable to a pair of compasses of infinite perfection. But, if so, the extremities of the imaginary instrument must be fixed at a distance apart equal to AB, after one of them has been accurately placed on the point A; and all this requires us to believe in the perception of an equality of two things without reference to any third thing. Again, when the two circles are described, and all their radii are determined by the interval between the compasses, this interval being equal to AB, the radii will be equal to AB; and as, of AB and AC, which are equal (in length), AB is equal (in length) to BC, therefore AC also is equal (in length) to BC, and the triangle which the three form is equilateral.

The same may be said of the superposition or application of one figure to another, had recourse to in the fourth proposition; it proceeds upon the assumption that the mind can intuitively perceive the equality of two things: otherwise there ought to be proof that AB and DE, AC and DF, respectively coincide throughout their whole length, just as BC and EF are proved to do. The definition of a straight line, as "that which lies evenly between its extreme points," does not provide for the coincidence of two straight lines in their intermediate points until their extreme points are shown to coincide. The practicability of it, however, is assumed, and we find little difficulty in making the concession. But the strict method of application of the triangle ABC to DEF would be first to place A accurately on D, allowing ABC to lie anywhere in the same plane as DEF, then to wheel ABC round so that B and C should describe circles with centre D (A) and radii AB (= DE), and AC (= DF) respectively. There would then be no doubt that E and F would be points on these circles, so that the application of B to E and C to F would be clearly practicable, and the remaining proof would present no difficulty. This may appear to some persons to be a case of hair-splitting; and it may be regarded as utterly useless to contend for the pre-eminence of one axiom over another, especially when those axioms are so closely related as the two we are considering, viz., "things which are equal to the same thing are equal to one another," and "things which are equal to one another are equal to the same thing." But it cannot be too strongly asserted that the mind starts, and does not merely conclude, with the equality of things, and that the intuitive perception by the mind of equality, and also of inequality, is the grand basis of reason. Now, no doubt, the idea may be regarded as implied in Euclid's axiom, when he says, "Things which are equal to the same thing;" but not only is there nothing to show how this equality is arrived at; on the contrary, the final clause of the axiom might indicate that the equality of things to one another, including of course that of each to the "same thing," was to be relegated back and back in an endless vista, its origin being lost in the dim distance, or in other words, that the mind always needed a third thing for the purpose of comparing any two. It is of importance to establish the fact that reason has its origin in intuitive perception, and this, I trust, has now been done.

The well-known symbols $A=B=C=D=$, &c., fairly represent the axiom that "the equality of equals to other things is coextensive."

There is another faculty of the mind, taken account of in the Differential and the Integral Calculus, and also to some extent in the Doctrine of Probabilities, viz., that faculty which gauges similarity where there is not absolute equality, and is content to ignore infinitesimal inequalities as making no practical difference in the work of life. But, when once these infinitesimal inequalities are discarded, the process is much the same as in the former case. Moreover, as a rule, such inequalities (relatively infinitesimal) can be diminished to any extent, though never entirely eliminated, except by arbitrary deletion.

To sum up, the mind takes cognizance of equality and of inequality, and so far gauges similarity as to accept as practically equal, things between which it perceives or concludes there is in the given respect only an infinitesimal inequality. Moreover, the mind can assume these things or take them for granted, where it cannot directly perceive them. And it proceeds from the equality or inequality of two things which it perceives, to the equality or inequality of these with other things, although it cannot directly compare and contrast them all. And this brings us to the greatest of all the laws of reason, viz., the Law of Substitution :—

Things which are equal in a given respect are in that respect equivalent, and may be substituted the one for the other, when the given respect only is concerned.

Thus, when it is said, "A pound sterling will cover the cost," we may with equal truth substitute for "A pound sterling," twenty shillings, or eight half-crowns.

This law governs most of the operations of Algebra and Geometry, where the given respect in which things are considered is simple and unambiguous, being generally that of number, extent, figure, and angular relation.

It also explains nearly all the "axioms" of Euclid. Thus, to take Axiom I. :—" Things which are equal to the same thing are equal to one another." Let $A = B$, then, if $B = C$, C can be substituted for B in the former equation and $A = C$.

Axiom II. :—" If equals be added to equals, the wholes are equal." $A+B = A+B$ being identical, but let $A = C$, and $B = D$, then by substitution $A+B = C+D$.

Axiom III. :—" If equals be taken from equals, the remainders are equal." $A-B = A-B$ being identical, but let $A = C$, and $B = D$, then $A-B = C-D$.

Axiom IV. :—" If equals be added to unequals, the wholes are unequal." Let $A\pm X = B$, and $C = D$; it is required to find the result of adding A to C and B to D. $B+C = B+C$ being identical, and by substitution $B+C = B+D = A\pm X+C$, so that $B+D$ is just so much greater or less than $A+C$, as B is greater or less than A, and the difference is represented by $\pm X$.

Axiom V. :—" If equals be taken from unequals, the remainders are unequal." Let $A\pm X = B$, and $C = D$; it is required to find the result of taking C from A, and D from B. $B-C = B-C$ being identical, and by substitution $B-C = B-D = A\pm X-C$, so that $B-D$ is just so much greater or less than $A-C$, as B is greater or less than A, and the difference is represented by $\pm X$.

Axiom VI. :—" Things which are double of the same are equal." Let $A = B$, then $A+A = A+A$ being identical, but by substitution

$A + A = B + B$, or $2A = 2B$. The same would hold good for any multiple.

Axiom VII. :—"Things which are halves of the same are equal to one another." This is more difficult to explain than any of the preceding, and is more like a self-evident truth incapable of logical proof. But definition will clear up much. The two halves of a thing are equal to each other, and are together equal to the whole. The doubles of each half are also equal (by Axiom VI.), and severally equal to the whole (by definition). And, of course, the law of substitution would allow us to substitute B for A on one side of the equation $\frac{1}{2}A = \frac{1}{2}A$, provided $A = B$. But the axiom before us involves the equality of the four quarters, and of the eight half-quarters, &c., of any single thing. One quarter might not be equal to another quarter, unless it were half of the same half. But $\frac{1}{2}A + \frac{1}{2}B = \frac{1}{2}A + \frac{1}{2}B$ being identical, and $\frac{1}{2}A + \frac{1}{2}B - B = \frac{1}{2}A + \frac{1}{2}B - A$ (Axiom III.); $\therefore \frac{1}{2}A - \frac{1}{2}B = \frac{1}{2}B - \frac{1}{2}A$, the two sides of which are identical with what would result if we assumed that $\frac{1}{2}A$ and $\frac{1}{2}B$ were equal and might be substituted for each other on one side of the undisputed equation $\frac{1}{2}A - \frac{1}{2}B = \frac{1}{2}A - \frac{1}{2}B$. Moreover, if $m - n$ can equal $n - m$, unless $m = n$, let us suppose that $n = m + x$; then $m - (m + x) = m + x - m$, that is $(m - m) - x = (m - m) + x$. Now no appreciable quantity can be added and subtracted in this way without making an appreciable difference; $\therefore x = 0$, and there is no difference between m and n, or between $\frac{1}{2}A$ and $\frac{1}{2}B$ in the cases instanced.

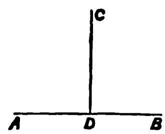
Axiom VIII. :—"Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another." This cannot be regarded as a distinct axiom, but as an application of Axiom I.; for things which fill the same space must be equal to the same thing. But the recognition of coincidence is of different kinds and degrees. Thus, in Book I., Prop. iv., when the one triangle is applied to the other, the coincidence of the point A with D is a matter, if not of perception, of pure assumption; but the coincidence of B with E is recognized only after an act of the reason; that of the lines AC and DF, and of the points C and F, only after further efforts of the reason; and, finally, that of the lines BC and EF, only after elaborate argument.

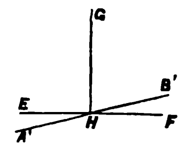
Axiom IX. :—"The whole is greater than its part." This again contains more of definition than of distinct axiomatic truth. For what is a *part* of anything? It is that which, if it be deducted from the whole, leaves another portion behind. Again, when is one thing *greater* than another? When it contains more. As Dr. Thomson, in his edition of Euclid, says, "the whole is equal to all its parts taken together"; for, if it be greater, it must contain something not forming a part of itself, and, if it be less, the parts must have increased spontaneously in size or number. Dr. Thomson simply defines a term, the *whole*.

Axiom X. :—"Two straight lines cannot enclose a space." Euclid's definition of a straight line is that of a *finite* straight line, since he says that it "lies evenly between its extreme points"; but in Postulate ii. he says, "Let it be granted, that a terminated straight line may be produced to any length in a straight line," and this gives us the idea of an *infinite* straight line. But do we require more than two infinite straight lines in one plane, each passing through certain given points in that plane, to determine a space? Or again, are we to consider the space between the lines to be unenclosed, when we assume the existence of two infinite straight lines parallel to each other, and do not further limit the figure? If an infernal spirit were a point moving in one plane, and his prison a

portion of that plane, bounded by two impassable straight lines, parallel to each other, and extended infinitely in both directions, it would take him a long time to gain his liberty; in fact, he would be practically shut in, not to say, "enclosed": he could never get to the extra-linear portions of the plane. It is evident, however, that Euclid, in the axiom before us, alludes to finite space and finite straight lines. But it is singular that he nowhere defines a "space," although he defines a *figure* as "that which is enclosed by one or more boundaries." We can scarcely regard the two as synonymous, but a limited space, defined by boundaries, is no doubt a figure. The simplest figures, bounded by lines alone, are on a *plane superficies*, and this is defined as "that in which, any two points being taken, the straight line between them lies wholly in that superficies." But it is of importance to observe that a *line* is "length without breadth," that a *superficies* is "that which hath only length and breadth," and that the latter is meant by Euclid when he says that two straight lines cannot enclose a "space." This is evident from the use he makes of the axiom in Prop. iv., where he supposes two straight lines to be drawn between the same two points, and adduces this axiom to prove that the two straight lines must needs coincide. But (though it is difficult to see how *space* can be enclosed on a plane surface) when once we know that the hypothetical "space" referred to has finite length and breadth, we need no distinct axiom to register the fact that two straight lines could not suffice for its boundaries; we simply require a proper definition of *breadth* as distinct from *length*. Let us suppose the length of a plane figure to be represented by a given straight line AB, then its breadth, if represented at all, must be represented by a straight line running in a direction transverse to AB. In this line (CD) an infinite number of points can be taken, to any one of which (E) straight lines can be drawn from A and B respectively; but AE and BE will not be in the same straight line (or AE produced will not reach B) unless E be in AB, and the "space" be thus eliminated. Much more is it clear that, if AC, AD, BC, BD be joined, (which will form the simplest rectilinear boundaries that can be conceived), more than two straight lines will result. Hence two straight lines cannot alone bound a figure. But this is proof, not axiom.

Axiom XI. :—"All right angles are equal to one another." This again is capable of proof. The definition of a right angle is as follows: "When a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it." Let us then take two sets of such lines and apply one to the other, so that the feet of the perpendiculars and the perpendiculars themselves may coincide. We can then prove that the remain-





ing lines must also coincide, or at least that the angle separating them is *nil.* Thus, let CD, AB and GH, EF be straight lines at right angles with each other. Apply D to H and let DC lie along HG; it is required to prove that AB will lie along EF, and that

the angle EHA′ or FHB′ will be *nil*. In the first place, if two straight lines, EF and A′B′, pass through the same point, H, they must either coincide or cut one another. This is as clear as many of the so-called axioms, but admits of proof. Let us on the one hand imagine that A′H can approach EF at an angle, and then coalesce with it along HF, A′HF and EHF, both remaining straight. This supposition is considered in a corollary to Prop. xi., Book I., but its absurdity can be shown at once; for, if we were required to produce the line FH in a backward direction Postulate ii. would be of no avail, unless it were further pointed out along which of the two, HE or HA′, it was to be continued; in fact, the line might proceed in any random direction. Let us next imagine that A′B, instead of cutting EF, can approach it at an angle during the first portion of its course, so as to touch it at H, and then recede from it *whilst still on the same side*. The absurdity of this conception is shown by taking into account, that the second portion of the line would, in receding from EF, reach (if produced) any given distance from it, and that a straight line parallel to EF and cutting A′H would thus meet that second portion, and two straight lines would enclose a space, cutting each other twice. Consequently, if the first portion of A′B′ be inclined at an angle to EF in somewhat the position of A′H, we are reduced to the necessity of regarding the second portion as lying in somewhat the position of HB′, that is, somewhere between the perpendicular GH and HF. But, by hypothesis, GHB′ = GHA′, and GHE = GHF, and, by the law of substitution, we have the following propositions:

$$\text{GHA}' > \text{GHE} \ (= \text{GHF}) > \text{GHB}' \ (= \text{GHA}').$$

Therefore, if A′B′ be inclined to EF so as to form appreciable angles, EHA′ and FHB′, we are driven to the absurd conclusion that a thing can be greater than itself. Hence we conclude that, applied as stated, AB would coincide with EF; and thus is it proved that all right angles are equal.*

Axiom XII.:—"If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles." Dr. Simson, admitting that this is not self-evident, demonstrates the truth of it by the aid of *five* different propositions! But it can be made clear on principles which are acknowledged to be valid at the very commencement of Book I. Superposition will aid us here as elsewhere. Let us suppose two parallel straight lines AB, CD to be placed at any distance apart, and also two sets of straight lines EF, GH,

* Lest it should be thought that the corollary to Prop. xi. alone justifies Postulate ii., I beg to state that Prop. xi. itself (from which the corollary is supposed to derive its stability) depends upon previous propositions in which Post. ii. forms an important feature. For Prop. viii., one of the direct assistants of Prop. xi., is dependent on Prop. vii., in which Post. ii. is used, and Prop. vii. is further dependent on Prop. v., in which Post. ii. is used. Moreover, Prop. iii., another of the direct assistants of Prop. xi., is dependent on Prop. ii., in which Post. ii. is used. Now, if Post. ii. depend for its justification on the corollary to Prop. xi., we are reasoning in a circle; and if, on the other hand, Post. ii. be granted on independent grounds, there is no need to postpone the corollary, to Prop. xi.

and IK, LM, in which (Fig. 2) equal angles are formed. Then, if

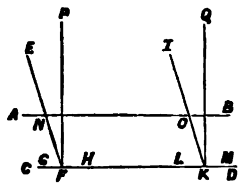


FIG. 2.

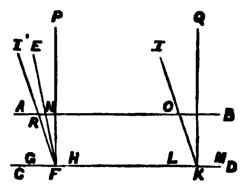


FIG. 3.

GH and LM be made to coincide with CD, as in the figure, $\angle$ IKC = EFC; and, if K be now made to approach F and coincide with it, then IK will coincide with EF, and both will cut AB in N. Let now IK, LM be withdrawn from this position, along CD, until they regain the position given in the figure. Is it not evident that every part of IK, LM is withdrawn from the corresponding part of EF, GH to an equal distance, and that accordingly ON = KF? And is it not also evident that this would be true, if FE and KI were produced indefinitely, and AB placed at any distance from CD? Such being the case, EF and KI, if produced, would never meet, but always maintain the same distance from each other; they are therefore parallel. Again, let PF, QK be imaginary perpendiculars to CD. Then IKC = EFC = PFC − PFE, and EFD = PFD + PFE; therefore, by addition, EFD + IKC = PFD + PFC, *i.e.*, the "two interior angles" = two right angles.

Let us next consider the case specially referred to in the "axiom," and in which the "two interior angles" EFD, IKC (Fig. 3) are less than two right angles. If we apply IK, LM to EF, GH now, IK will not, as before, coincide with EF, but take up the position indicated by I'F; that is $\angle$ I'FC will be less than EFC, for I'FC = IKC, and, by the hypothesis, EFD + IKC are less than two right angles, therefore I'FC + EFD are less than PFD + PFC, not greater, as they would have to be to place I'F on the other side of EF. ON, therefore, on AB, is less than OR, which is the distance (measured in a direction parallel to CD) between the parallels I'F, IK; and, as RN must needs widen, the more remote AB is from CD, it will gradually encroach upon the constant OR, until ON is reduced to a point by the meeting of FE and KI produced.

We have thus considered, in detail, the twelve Axioms of Euclid, Book I. The axioms prefixed to some of the succeeding books are all seen to be amenable to the Law of Substitution, without the slightest difficulty.

And, step by step, this Law of Substitution can bring us to the highest pinnacles of mathematical knowledge, all that is requisite being a clear definition and perception of terms, and a careful inspection, lest a step should be taken for which the law gives no warrant.

But when we leave the domain of mathematics, where the data, being of an abstract character, can be accurately defined, and where, assuming that A = B, it is also true that B = A, and when we come down to the Logic of Common Life, where it is well-nigh impossible to declare the absolute equality of any two things in any one respect, where an element of doubt clings to all our data, where in consequence of proceeding from particulars to generals, instead of from generals to particulars, we can rarely arrive at a universal proposition, and where, even granting that "all A is

B," we can only deduce therefrom that "some B is A,"—although the Law of Substitution still holds good, we need additional safeguards against error.

Still, as Whately says, "the rules of Logic have nothing to do with the truth or falsity of the premises, but merely teach us to decide (not whether the premises are *fairly laid down*, but) whether the conclusion *follows fairly from* the premises or not. The *degree of evidence* for any proposition we originally assume as a premiss, is not to be learned from Logic, but is the province of whatever science furnishes the subject-matter of the argument. *E.g.*, from examination of many horned animals, as sheep, cows, &c., a Naturalist finds that they have cloven feet; now *his skill as a Naturalist* is to be shown in judging whether these animals are likely to resemble in the form of their feet all other horned animals; and it is the exercise of this judgment, together with the examination of individuals, that constitutes what is usually meant by the *Inductive Process*, which is that by which we gain *new truths*, and which is not connected with Logic, being not what is strictly called *Reasoning*, but *Investigation*."

It is essential, however, as Whately further says, "that we should abstract that portion of any object presented to the mind which is important to the argument in hand. There are expressions in common use which have a reference to this caution; such as, 'this is a question, not as to the nature of the object, but the magnitude of it,' 'this is a question of time, or of place, &c.'; that is, 'the subject must be referred to this or to that category.' The categories enumerated by Aristotle, are *οὐσία* (being, essence), *πόσον* (how great?), *ποῖον* (of what kind?), *πρός τι* (toward what?), *ποῦ* (where?), *πότε* (when?), *κεῖσθαι* (is lying?), *ἔχειν* (is holding?), *ποιεῖν* (is doing?), *πάσχειν* (is suffering?)—which are usually rendered, as adequately as perhaps they can be in our language, Substance, Quantity, Quality, Relation, Place, Time, Situation, Possession, Action, Suffering, and may all be ultimately referred to the two heads of *Substance* and *Attribute* (or Accident)."

But when care is taken to consider things in one definite respect at a time, so as to compare and contrast properly, much help may be derived from the following method of classification:—

$$\text{A}\left\{\begin{array}{l}\text{B, or A non Z}\left\{\begin{array}{l}\text{C, or AB non Y,}\\ \text{Y, or AB non C,}\end{array}\right.\\ \left.\begin{array}{l}\text{Z, or A non B}\\ \text{W, or X non Z}\end{array}\right\}\text{X.}\end{array}\right.$$

Let A be the most comprehensive class or order of things referred to in the argument, whether as subject or predicate, under one given aspect; and let A comprise under it two genera only, B and Z, which are mutually exclusive; also let B comprise under it two species (or individual and remainder) C and Y, mutually exclusive. Moreover, let Z, which may contain any number of individuals from one upwards, provided they be A non B's, be wholly comprehended under a different order of things, X, so that the thing or things indicated can be referred to as AXZ or XAZ, as most convenient; and let W be the remainder of X. We can now illustrate the nineteen legitimate forms of Syllogism (including

twelve different *moods*), under the four divisions (called *figures*) defined by the position of the "middle term."

I. 1. *Barbara.*—All B's are A's; all C's are B's; ∴ all C's are A's.

I. 2. *Celarent.*—No B's are Z's; all Y's are B's; ∴ no Y's are Z's.

I. 3. *Darii.*—All Z's are X's; some A's are Z's; ∴ some A's are X's.

I. 4. *Ferio.*—No Z's are B's; some X's are Z's; ∴ some X's are not B's.

N.B.—It is important here to note that B and W are not mutually exclusive, nor yet co-extensive.

II. 1. *Cesare.*—No Z's are B's; all C's are B's; ∴ no C's are Z's.

II. 2. *Camestres.*—All Y's are B's; no Z's are B's; ∴ no Z's are Y's.

II. 3. *Festino.*—No Y is C; some B's are C; ∴ some B's are not Y's.

II. 4. *Baroko.*—All C's are B's; some A's are not B's; ∴ some A's are not C's.

III. 1. *Darapti.*—All Z's are A's; all Z's are X's; ∴ some X's are A's.

III. 2. *Disamis.*—Some B's are Y; all B's are A's; ∴ some A's are Y.

III. 3. *Datisi.*—All B's are A's; some B's are C's; ∴ some C's are A's.

N.B.—This conclusion is particular where it might have been universal.

III. 4. *Felapton.*—No Z's are W's; all Z's are A's; ∴ some A's are not W's.

N.B.—This does not assert that no W's are A's.

III. 5. *Bokardo.*—Some B's are not Y's; all B's are A's; ∴ some A's are not Y's.

III. 6. *Feriso.*—No B's are Z's; some B's are Y's; ∴ some Y's are not Z's.

N.B.—This again is particular where it might be universal; but the minor proposition might have been (within the truth) "some B's are A's"; and then "some A's are not Z's" could not have been made universal.

IV. 1. *Bramantip.*—All C's are B's; all B's are A's; ∴ some A's are C's.

IV. 2. *Camenes.*—All C's are B's; no B's are Z's; ∴ no Z's are C's.

IV. 3. *Dimaris.*—Some A's are Z's; all Z's are X's; ∴ some X's are A's.

IV. 4. *Felapo.*—No W's are Z's; all Z's are A's; ∴ some A's are not W's.

IV. 5. *Fresison.*—No Z's are B's; some B's are C's; ∴ some C's are not Z's.

N.B.—The same remarks apply here as after III. 6.

Before proceeding further, I may observe that the relation of Z to X is a most important one. Z may be selected from the A's, not because it belongs to the X's, but on other grounds, such as for the purpose of experiment, or even by accident; but having been selected, and possibly subjected to a particular treatment, it proves to belong to a distinct order of things, and the class X is founded. Other A's being similarly examined or tested, may or may not produce the same result, but, in any case, fresh knowledge is acquired, a discovery is made leading to experiment in various directions, and possibly an invention is originated. Thus are Logic and Induction seen to be very closely related, if not identical processes.

But, of course, X will equally serve to represent in other instances the quality, or class of qualities, to which Z's owe their distinction from B's, or any other quality or attribute which Z's possess in addition to that indicated by A.

A few instances of reasoning conducted by the method now outlined, will not be out of place. There is the famous discussion, referred to by Whately, regarding *Prudence*, dating from the time of Aristotle, and continued in later days by Hutcheson and Adam Smith. Hutcheson placed all virtue in benevolence; but Adam Smith replied in terms of the following syllogism:—Prudence has for its object the benefit of the individual; prudence is a virtue; therefore some virtue has for its object the benefit of the individual.

Let X = virtues, and A = things having for their object the benefit of the individual. Then, if it be granted that Prudence is XAZ, or one of the virtues having for its object the benefit of the individual, Adam Smith's conclusion follows, by *Darapti* (III., 1).

But Aristotle wished to prove the virtues inseparable (*Ethics*, Book VI.), and made use of the following singular syllogism, which is employed by Whately to illustrate the first and easiest mood (I., 1):—"He who possesses prudence, possesses all virtue; he who possesses one virtue, must possess prudence; therefore, he who possesses one, possesses all."

Let A = the possessors of at least one virtue, B = the possessors of prudence, and C = the possessors of all the virtues. This is the natural classification of the three terms, and that which, *primâ facie*, we are bound to make; for, until we know that the virtues are inseparable, and not variously distributed, we must perforce assume that persons of whom it may simply be said that they are not without any virtue, will form the more comprehensive class, and those having all the virtues, the more select. But the Stagirite reverses the order of things, and those having all the virtues are made to include those having one. The syllogism is thus made to stand on its head instead of its feet; and the conclusion certainly cannot be maintained, unless, with the natural classification given above, proof can be given that AB non C's and A non B's have no existence. But is it true that there are no prudent persons except those who have all the virtues? Is it true that those who have not prudence, have no virtue? According to Aristotle, the class A is (so far as virtue is concerned) indivisible, so that, if we be convinced that a man has one virtue, we may safely take the remainder of his nature on trust! A strange *prudence* this! It is true, no doubt, that the word has had many meanings. Thus, according to Paley, some philosophers have held that *Benevolence* proposes good ends, *Prudence* suggests the best means of attaining them, while *Fortitude* and *Temperance* complete the list of virtues. But this makes prudence to be mere practical wisdom, a matter of the intellect. "Virtue is distinguished by others," says Paley again, "into two branches only, *prudence* and *benevolence*—prudence, attentive to our own interest; benevolence, to that of our fellow creatures." But if this, agreeing with Adam Smith's definition (given above), be correct, there have been many noble and apparently virtuous deeds, from which the suspicion of prudential motives would have seriously detracted. If it be a virtue to risk life, and to encounter death itself, for the sake of another, in what consists the prudence, unless it be in what Butler calls "the principle of reasonable and *cool* self-love"? Are there no beings who entirely forget themselves in their anxiety for others' good? Or, if such exist, must we take from them their patent of nobility as the supremely virtuous? Human nature, as at present constituted, is far from perfect, and, even in the best men, the virtues are by no means evenly balanced. But the man who errs on the side of Benevolence and Self-sacrifice, is surely to be preferred to one who errs on the side of self-

interest, or even of practical wisdom; for there is such a thing as cultivating prudence to the destruction of the other virtues. Such being the case, it seems monstrous to consider prudence as the essential virtue without which no other can exist. A man who has one virtue may have prudence, but the man (of the A non B's) who is imprudently brave in a good cause is of a much nobler sort. Now, a well-grounded suspicion that even one A is not B, or one AB not C, that is, that a man not devoid of virtue may be imprudent, or a prudent man not wholly virtuous, is sufficient to upset an argument of such extremely unstable equilibrium as that we have considered.

Our next instance shall be taken from Paley's celebrated disquisition on a *moral sense*, its existence in man or otherwise, given in the work on Moral Philosophy (Book I., Chap. 5). He begins by giving the case of Caius Toranius, who betrayed his own father to arrest and death; and, after depicting the deed in all its malignity, he says, "The question is, whether, if this story were related to the wild boy caught some years ago in the woods of Hanover, or to a savage without experience, and without instruction, cut off in his infancy from all intercourse with his species, and, consequently, under no possible influence of example, authority, education, sympathy, or habit; whether such a one would feel, upon the relation, any degree of that sentiment of disapprobation of Toranius' conduct which we feel or not?" And that we may be in no doubt as to what he considers to be the matter in dispute, he further says, "They who maintain the existence of a moral sense; of innate maxims; of a natural conscience; that the love of virtue and the hatred of vice are instinctive, or the perception of right and wrong intuitive (all which are only different ways of expressing the same opinion), affirm that he would. They who deny the existence of a moral sense, &c., affirm that he would not." After saying that "what would be the event can only be judged of from probable reasons," he proceeds in the most lucid language to give the various reasons adduced on either side. Thus, the one party assert that a certain approbation of noble deeds and a corresponding condemnation of vice, are instantaneous and without deliberation; and also uniform and universal. But the other side show that nearly every form of vice has at some time or in some country been countenanced by public opinion, even by philosophers and others in high position; that we ourselves do not perfectly agree as to what is right and what is wrong; and that the general though not universal approval of certain lines of conduct may be accounted for in various ways. For instance, "having experienced at some time, a particular conduct to be beneficial to ourselves, or observed that it would be so, a sentiment of approbation rises up in our minds, which sentiment afterwards accompanies the idea or mention of the same conduct, although the private advantage which first excited it no longer exist." By these means the custom of approving certain actions *commenced*: it is kept up by authority, by imitation, by inculcation, by habit. Besides, say they, none of the so-called *innate maxims* are absolutely and universally true, but all *bend* to circumstances. Thus, veracity, which seems, if any be, a natural duty, is excused in many cases towards an enemy, a thief, or a madman; and so with the obligation to keep a promise. Nothing is so soon made as a maxim: Aristotle laid down, as a fundamental and self-evident maxim, that nature intended barbarians to be slaves. "Upon the whole," says Paley, "it seems to me, either that there exist no such instincts as compose what is called the moral sense, or that they are not now to be distinguished from

prejudices and habits; on which account they cannot be depended upon in moral reasoning; that is, it is not a safe way of arguing, to assume certain principles as so many dictates, impulses, and instincts of nature, and then to draw conclusions from these principles, as to the rectitude or wrongness of actions, independent of the tendency of such actions, or of any other considerations whatever"; and he finishes by dismissing the question as of no concern except to the curious.

But a very different complexion is put upon the matter by a careful classification of the chief terms. *Morals* may be divided into our own and other people's, and under both these heads we may place on one side *overt acts, habits*, &c., and on the other side, what are summed up under the designation of *motives*—those secret springs of thought and action which may be inferred, but cannot be perceived, by outsiders. These motives act in the higher regions of the being's nature, in those parts which are in immediate relation with the sentient power, and they produce an impression, agreeable or otherwise, according to their harmony or discord with what the being himself accepts as *right*. As the rain-drops descend upon the sides of a mountain, and, percolating through the several strata, reach the central reservoir whence the streams receive their supply, and as the set of the strata determines in great measure the particular side of the mountain on which the spring will appear, so a man's deeds are the resultants of the various influences brought to bear upon him, and, in his reaction upon the outer world, he is able, by his Will, to determine more or less the character of his acts. It is at this juncture that the conscience comes in, its province being to perceive the equality or inequality of a nascent act to the being's accepted standard of right, that is, to the degree of *light* he possesses. If, at the critical moment, temptation prevail, a painful impression is produced, but, if the temptation be withstood and overcome, the result is pleasing. In these respects the moral sense is like the other senses, which perceive equality or inequality in things which concern them, and produce corresponding impressions. But the conscience or moral sense of one man is not concerned with the overt acts, much less the motives, of another man. The overt acts of others may be judged of by the Reason, and, if good, followed, if bad, shunned; but it must not be forgotten that what is good, or at least harmless, for one man, may be extremely blameworthy in another. The rules that suit everybody are broad indeed. Caius Toranius may have been, and probably was, the greatest blackguard imaginable; but to reprobate his conduct will not mend matters for me. The question for *my* conscience is, how far *my* present conduct tallies with *my* present light. Moreover, the moral sense can be blunted and destroyed, or educated and refined, much as any other. This and various circumstances concur to produce at different times, and in different localities, habits and customs which differ greatly on the score of morality. But to deny the existence of a moral sense on this account, is like denying the sense of hearing, because the accepted music of one nation is discord and confusion to another; or like denying the sense of sight, because one man beholds beauty where another sees only so much canvas and paint.

There are cases in which we can so divide any class of things under consideration, as to give the exact or approximate proportion borne by the particular genus or species selected, to the class of which it forms part. Scientific observations and the results of experiments often admit of careful division of this sort; and thus some idea is gained of the relative importance of events. The Method of Classification now advocated is also

useful in the Calculation of Chances, especially in cases of a mixed character. The following instance, by way of illustration, is from Boole's "Laws of Thought":—"The chances of two causes A_1 and A_2 are c_1 and c_2 respectively. The chance that, if the cause A_1 present itself, an event E will accompany it, whether as a consequence of the cause A_1 or not, is p_1; and the chance that, if the cause A_2 present itself, the event E will accompany it, whether as a consequence of it or not, is p_2. Moreover, the event E cannot appear in the absence of both the causes A_1 and A_2. Required the chance of the event E." Here we may leave out the things signified, and merely put the proportions; and then, taking unity as the class-total for each cause, under which to put c_1 or c_2, the probability of its occurrence, and $(1-c_1)$ or $(1-c_2)$, the probability of its non-occurrence; and, dividing these probabilities again, according to the question, we obtain the following scheme:—

$$1\left\{\begin{array}{l}(1-c_1)\\ \\ c_1\cdots\cdots\left\{\begin{array}{l}c_1p_1\cdots\cdots\left\{\begin{array}{l}y\\ z\end{array}\right.\\ c_1(1-p_1)\end{array}\right.\end{array}\right. \qquad \left.\begin{array}{l}x\\ y\end{array}\right\}\cdots\cdots p_2c_2 \qquad \left.\begin{array}{r}(1-c_2)\\ \\ \left.\begin{array}{r}p_2c_2\\ (1-p_2)c_2\end{array}\right\}\cdots\cdots c_2\end{array}\right\}1$$

in which z = chance that E will occur when A_1 alone is present, x = chance when A_2 alone is present, y = chance when both A_1 and A_2 are present. Let u = total chance = $x+y+z$. Now, in the question there is nothing to show how E is affected by the combination of the two causes. Consequently, y must be regarded as a variable quantity, and the other two as varying more or less with it. If E occurs only when A_1 and A_2 are present separately, then $y=0$; if one of the causes be inoperative unless the other be present, then either $x=0$, or $z=0$, but this cannot be the case with both, unless $c_1p_1 = c_2p_2$. In any case, $u+y = c_1p_1+c_2p_2$. If we assume that E is inevitable when both causes are present, then $y = c_1c_2$, and $u = c_1p_1+c_2p_2-c_1c_2$. But if we assume that the occurrence of E, when both causes are present, simply bears to its total occurrence the same proportion that their combined occurrence bears to their total occurrence, then $y:u = c_1c_2 : c_1+c_2-c_1c_2$ and $u = (c_1p_1+c_2p_2)\left(1-\frac{c_1c_2}{c_1+c_2}\right)$. If we take $c_1 = \cdot 1$, $c_2 = \cdot 2$, $p_1 = \cdot 6$, and $p_2 = \cdot 7$, we shall find that, under the former assumption, $u = \cdot 18$, and, under the latter assumption, $u = \cdot 18'6'$. Professor Boole's equation for finding the *exact* value of the required chance (based apparently upon the assumption that A_1 and A_2 are independent, but that E is very little more probable when both are present) is as follows:—

$$\frac{(u-c_1p_1)\,(u-c_2p_2)}{c_1p_1+c_2p_2-u} = \frac{\{1-c_1(1-p_1)-u\}\,\{1-c_2(1-p_2)-u\}}{1-u},$$

which, translated to suit our present scheme, is as follows:—

$$\frac{xz}{y} = \frac{(1-c_1-x)\,(1-c_2-z)}{1-u},$$

showing that the non-occurrences play a prominent part in its formation, a fact rather difficult to account for. Giving the above numerical values to the terms of Professor Boole's equation, we find that $u = \cdot 19068$. This nearly approaches the result obtained by the following abstract ratio:—

$y : u = c_1p_1c_2p_2 : c_1p_1 + c_2p_2 - c_1p_1c_2p_2$, whence $y = c_1p_1c_2p_2$, and $u = \cdot 1916$, in which case the two causes might be regarded as totally unhelpful of each other; for, under the conditions, the chance of E occurring rises the nearer y approaches to zero, and $c_1p_1\,c_2p_2$ would represent the lowest assignable value of y, unless special circumstances were known to exist rendering the combination of A_1 and A_2 impossible.

Another instance may be taken from the same source (Boole's "Laws of Thought"). "The chance that a witness A speaks the truth is p, the chance that another witness B speaks the truth is q, and the chance that they disagree in a statement is r. What is the chance that, if they agree, their statement is true?" Here taking A and B to restrict their answers to *Yes* or *No*, or to give what is equivalent to an affirmation or denial of a statement submitted to both alike, we can easily produce the following scheme:—

$$1\left\{\begin{array}{l}(1-p)\left\{\begin{array}{l} z=w\\ y\end{array}\right.\\[1em] p\ \ldots\ldots\left\{\begin{array}{l}x\\ u\\ z=w\end{array}\right.\end{array}\right. \qquad \left.\begin{array}{l}\left.\begin{array}{l}y\\ x\end{array}\right\}\ldots(1-q)\\[1em] \left.\begin{array}{l}u\\ z=w\end{array}\right\}\ldots\ldots\ldots\ q\end{array}\right\}1$$

in which u = chance that A and B will agree truthfully, and y = chance that they will agree falsely. Therefore $u+y = 1-r$, and $\frac{u}{u+y} = \frac{u}{1-r}$ = chance required. Now $x+y = 1-q$, $\therefore u-x = q-r$, but $u+x = p$, $\therefore u = \frac{p+q-r}{2}$, and chance required $= \frac{p+q-r}{2\,(1-r)}$. Of course, if A and B are wholly independent (which is not assumed in the question), $u = pq$, and $y = (1-p)(1-q)$, and the chance required $= \frac{pq}{pq+(1-p)(1-q)}$. If, therefore, we have had an opportunity of gauging p and q, before the present series of questions is put to A and B, we are able to estimate the degree of suspicion which should attach to their present answers on the score of collusion, by noting during the series how much $(1-r)$ exceeds $\{pq+(1-p)(1-q)\}$.

A further instance may be taken from the *Educational Times* for Dec. 1884, Question 7957 (by Rev. T. C. Simmons, M.A.):—"Solve the equations $x^2-yz = a^2$, $y^2-zx = b^2$, $z^2-xy = c^2$." This is simplified as follows:—

$$(x+y+z)^2\left\{\begin{array}{l} x\,(x+y+z)\left\{\begin{array}{l}x^2=x^2\\ xy=z^2-c^2\\ xz=y^2-b^2\end{array}\right.\\[1em] y\,(x+y+z)\left\{\begin{array}{l}xy=z^2-c^2\\ y^2=y^2\\ yz=x^2-a^2\end{array}\right.\\[1em] z\,(x+y+z)\left\{\begin{array}{l}xz=y^2-b^2\\ yz=x^2-a^2\\ z^2=z^2\end{array}\right.\end{array}\right.$$

Then
$$x^2+y^2+z^2-b^2-c^2 = x(x+y+z),$$
$$x^2+y^2+z^2-a^2-c^2 = y(x+y+z),$$
$$x^2+y^2+z^2-a^2-b^2 = z(x+y+z),$$
and
$$x(x+y+z)-a^2 = y(x+y+z)-b^2 = z(x+y+z)-c^2.$$
Let $x+y+z = \mathrm{S}$, and $u = \mathrm{S}x-a^2 = \mathrm{S}y-b^2 = \mathrm{S}z-c^2$; then, by addition,
$\mathrm{S}^2-3u = a^2+b^2+c^2$, and $u = \dfrac{\mathrm{S}^2-(a^2+b^2+c^2)}{3}$.

Therefore
$$x = \frac{u+a^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - a^2\right),$$
$$y = \frac{u+b^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - b^2\right),$$
$$z = \frac{u+c^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - c^2\right),$$
and, a^2, b^2, c^2 being constants, S is a variable, and x, y, z vary with it, forming the coordinates of a curve which is not in one plane. But, as S increases, x, y, z approximate to equality, for $y = x - \dfrac{a^2-b^2}{\mathrm{S}}$, and $z = x - \dfrac{a^2-c^2}{\mathrm{S}}$.

In conclusion, I cannot do better than advise those who wish to be carried safely and expeditiously through a long argument, or through certain calculations such as those preceding Multiple Integration, to use Mr. Hugh McColl's System of Notation, which he calls, "The Calculus of Equivalent Statements." It is highly ingenious, yet very simple, and needs only to be gathered under one cover, to be very widely used. At present, its head, trunk, and limbs lie scattered in different publications, of which the chief are as follows:—An article on "Symbolical Reasoning" in *Mind* (No. 17, Jan. 1880); "On the growth and use of a Symbolical Language" in the *Memoirs of the Manchester Literary and Philosophical Society* (1880—81); and four papers which have appeared at various times in the *Proceedings of the London Mathematical Society* (Vol. ix., Nos. 122, 123, read Nov., 1877; No. 135, read June, 1878; Vol. x., Nos. 141, 142; and Vol. xi., No. 163). But one of the earliest papers was published in the *Educational Times* (Vol. xxviii., p. 20). *The London Mathematical Society* has other papers under consideration at the present time from the same hand, and which it is to be hoped will soon be forthcoming.

London: C. F. Hodgson & Son, Printers, Gough Square, Fleet Street, E.C.

74